"Production"

Reynolds

I. Production is the process of altering or transforming "inputs" to increase their ability to satisfy wants.

A. Changes include:

1. **a change in physical characteristics** (The cooking of hamburger meat increases its ability to satisfy my wants. The meat has been physically altered.)
2. **a change in location** (Moving lobster from Maine to Idaho increases the satisfaction of wants. The consumers in Idaho are willing (and able) to pay more for lobster than someone in Maine. If they weren't, the lobster would stay in Maine.)
3. **a change in time a good** (or service) **is available** (If I had to eat all the meat from a animal that was just butchered before the meat spoiled, it would give less utility than if it could be frozen and used over a longer time period.)
4. **a change in ownership** (a broker that gets a buyer and seller together provides a "productive" service.)

B. Inputs include:

- Land (R or N) ("Gifts of Nature" or natural resources from the natural environment. Some natural resources are flow resources (like sun, wind). Others are "renewable" and some are "stocks." In some cases a resource may be considered "exhaustible."
- Labour (**L**)(Human effort (physical or mental) used to produce goods and services.)
- Kapital (**K**)(Inputs that are goods that were produced to be used for the further production of goods and services. Money can be used to buy kapital but is not itself kapital.)
- Entrepreneurial Ability (This input was added to deal with the problems created by Euler's Theorem and to justify the allocation of the "surplus.")

1. Inputs can be difficult to measure.
2. The classification of inputs is not always straightforward. Is a stand of timber that has been planted a kapital good or a natural resource?
3. Frequently inputs are measured in monetary terms.

II. Production and Costs

The costs of production are determined by the production function and the prices of inputs. The production function describes the relationship between a set of inputs, a state of technology and the output. It describes physical relationships between inputs and outputs. The cost functions describe the pecuniary or financial relationships between inputs and output and the costs of production. An understanding of production relationships is an aid to interpreting cost data.

III. Plants, Firms and Industries

A plant is a physical unit of production. It typically consists of buildings, equipment and other physical inputs. These inputs can be measured in a variety of ways. Both cost and production data will often relate to a plant that is described by some fixed input and produces a single homogeneous output. Joint products and joint costs result in problems of allocating costs.

A firm is an organizational structure that manages the operations of the units of production or plants. A firm may have no physical plants (Nike) where the production of goods takes place or they may have many plants.

An industry is a group of firms who produce a similar product. The auto or steel industries are examples.

IV. The Production Function

The production function is a model that specifies the relationship between a set of inputs and a set of outputs. It is "like" a recipe in a cookbook. A production function can be expressed as:

$$Q = f(L, K, R, \text{technology}, \ldots)$$

Certainly, social institutions can influence production decisions and outcomes. Laws and regulations are examples of social institutions that may constrain production choices.

A. Time Periods

According to Kurt Vonnegut, "Time is mother nature's way of preventing everything from happening at once." In economics (and other fields like accounting and finance), time is complicated. Dynamic models are far more difficult than static models, so comparative statics is the typical approach the problem of time in economics (accounting and finance). Time is divided into segments. Accounting data is often considered as short-run or long run. In economics, there are four time periods used; the market period, short-run, long-run and very long run.

B. Market Period

The market period is defined as a period of time so short that technology used is constant and the producer(s) cannot alter any inputs or output. A fishing ship that has returned and docked has its "catch" that cannot be altered with in the market period. They cannot alter the inputs or the output of the production process. The output of fish must be sold or disposed of in some way. The price of the fish may be high or low but the inputs or output cannot be altered.
Quantity as well as inputs is fixed so there are no interesting questions about production, rather the concern is about the price of the output.

C. Short-run Period

The short-run is a period of time long enough for the producer(s) to change at least one input while other inputs and technology are constant. In the simple case there is one homogeneous input that is added to a given "scale of plant" (defined by a fixed input) and one output. In many cases, the "size" of a plant or the "scale" of plant is defined and described by the fixed input. The size of a farm may be described by the number of acres under cultivation; the size of a hair dressers by the number of chairs or stations; the size of a warehouse by square feet of storage space.

In the short-run the production function can be stated:

$Q = f(L)$, ceteris paribus; where Q is output and L is the variable input (in this case labour.

D. Long-run Period

The long-run is a period of such duration that all inputs can be changed although technology is constant. The producer(s) can alter their "scale" of plant.

In the long-run the production function might be stated:

$Q = f(L, K, R)$, ceteris paribus

Where L is labour, K is capital and R is natural resources

If the production process is limited to two inputs (so we can develop simple three dimensional models that can be described in two dimensional space) it can be stated:

Q = f(L, K), ceteris paribus

The analysis will begin with a two input model that shows the production of a single output.

E. **Very Long-run Period**

A period of time of sufficient duration that technology as well as all inputs can change is described as the "very long-run." The very long run presents special problems for analysis.

V. Estimation of Production Functions

There are several ways that decision makers may "know" and use production functions. Any decision maker must determine the scale of plant (or size of the fixed input) and the amount of variable input to use. To make these decisions, they must know the relationship between changes in inputs and changes in output.

A. In some cases decision makers may "intuitively" know the effects of a change in an input. A farmer who has experience with growing corn may know when the corn needs fertilizer of water.

B. Industrial engineers use engineering data to design production systems. If someone were designing a brewery to produce 1000 gallons of beer per week they would know the volume of liquid needed, the inputs of hops, barley, sugar, etc and the sizes of the vats, pumps, hoses, burners and energy to process the beer.

C. Statistical models of production can be done in some cases. A feedlot that processes beef may collect data on diets fed to the animals and weight gain. For each weight class it may be possible to correlate certain feeds and weight gain.

D. If cost data is known and you can control for changes in conditions it may be possible to estimate production relationships.

VI. Long-run Production Model

In the long-run technology is held constant but all inputs and output can be altered. In our model we will consider two inputs (L and K) and a single output (Q_X). Production functions can be structured in a variety of ways but the simplest version is the Cobb-Douglas production function.

A. The Cobb-Douglas Production Function

The Cobb-Douglas production function is a special case of the "constant elasticity of substitution" production function. It has several attributes that are useful in describing the production relationship. However, it fails to show the changes in a production relationship that changes from increasing to constant to decreasing returns to scale. According to Murray Brown (in the *New Palgrave*, p 461) the "Cobb-Douglas from is remarkably robust in a vast variety of applications and that it will endure is hardly in question."

The Cobb-Douglas is often stated in the long run form:

Q_X = f(L, K),given technology. More specifically, it is

$$Q_X = AL^{\alpha}K^{\beta}$$

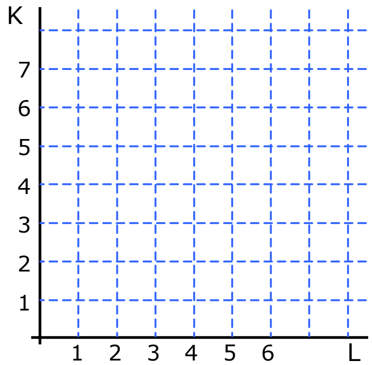
Q_X = output
A = an "efficiency parameter" and relates to technology
L = units of labour
K = units of kapital

α and β reflect labour and kapital's contribution to output and may be altered by technological change.

In this form, the Cobb-Douglas does not show a production relationship that changes from increasing, constant and decreasing returns over different levels of production. If $\alpha + \beta = 1$,there are "constant returns" to scale.

If $\alpha + \beta > 1$, there will be "increasing returns" and when $\alpha + \beta < 1$, there will be "decreasing returns" in production.

Figure 1



B. A Production Surface

In the long-run both labour (L) and kaptial (K) are variable as well as output. Three variables requires a three dimensional model. However, like utility maps and indifference curves, three-dimensional models can be projected onto a two dimensional graph.

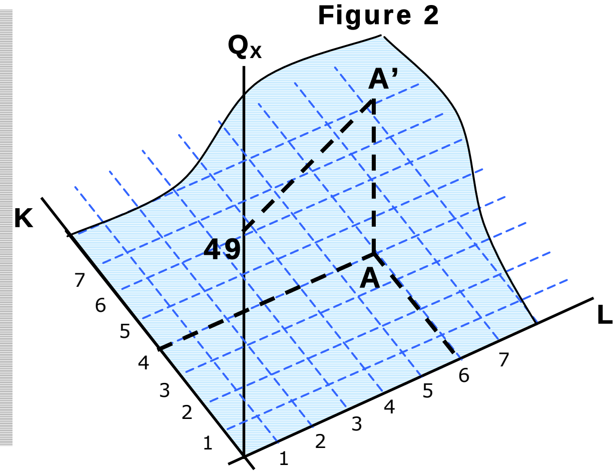
Imagine a corner of a room with a pile of sand in the room. Along one baseboard, labour (L) is measured while kapital (K) is measured along the other. Output (Q_X) is measured along the corner between the two axes, L and K. The elevation is Q_X. For any L and any K there is a coordinate on the floor of the room. In Figure 1, for any L and any K there is one coordinate. The elevation of the surface of the sand pile at that coordinate is a measure of output. The three axes for L, K and Q_X (output) are shown in Figure 2. Labour is measured along the right axis, Kapital along the left and output (Q_X) along the vertical axis.

In Figure 2, labour is measured along the axis on the right, Kapital on the left and the quantity (Q_X) is on the vertical.

In the example, when labour is 6 and K is 4 the coordinate (6,4) is at point A. At the coordinate A, the elevation or output is 49 and is measured along the vertical axis.

For any L and K the elevation of the production surface can be measured along the vertical axis.

Figure 2



C. A Production Matrix

The information in Figure 2 can be shown as a matrix. In Figure 3 the Cobb-Douglass Production $Q_X = 10\ L^{.5}\ K^{.5}$ is shown as a matrix. Both L and K are shown in discrete units. The elevation of the production surface (Q_X) shown in Figure 2 is shown as discrete data in the cells identified by an amount of labour and kapital.

Labour is shown along the bottom of the matrix in discrete units. The value of L ranges from 0 to 17 units. Kapital is shown along the vertical, right hand side of the matrix and ranges from 0 to 20 units. In each cell the amount of output (or elevation of the production surface) associated with each combination of Labour and Kapital is shown. When L=6 and K=4, the output is 48.99 units of good X. What is the output when 10 units of Labour and 13 units of Kapital are used?

K	**20**	0	44.721	63.246	77.46	89.443	100	109.54	118.32	126.49	134.16	141.42	148.32	154.92	161.25	167.33	173.21	178.89	184.39
A	**19**	0	43.589	61.644	75.498	87.178	97.468	106.77	115.33	123.29	130.77	137.84	144.57	151	157.16	163.1	168.82	174.36	179.72
P	**18**	0	42.426	60	73.485	84.853	94.868	103.92	112.25	120	127.28	134.16	140.71	146.97	152.97	158.75	164.32	169.71	174.93
I	**17**	0	41.231	58.31	71.414	82.462	92.195	101	109.09	116.62	123.69	130.38	136.75	142.83	148.66	154.27	159.69	164.92	170
T	**16**	0	40	56.569	69.282	80	89.443	97.98	105.83	113.14	120	126.49	132.66	138.56	144.22	149.67	154.92	160	164.92
A	**15**	0	38.73	54.772	67.082	77.46	86.603	94.868	102.47	109.54	116.19	122.47	128.45	134.16	139.64	144.91	150	154.92	159.69
L	**14**	0	37.417	52.915	64.807	74.833	83.666	91.652	98.995	105.83	112.25	118.32	124.1	129.61	134.91	140	144.91	149.67	154.27
	13	0	36.056	50.99	62.45	72.111	80.623	88.318	95.394	101.98	108.17	114.02	119.58	124.9	130	134.91	139.64	144.22	148.66
	12	0	34.641	48.99	60	69.282	77.46	84.853	91.652	97.98	103.92	109.54	114.89	120	124.9	129.61	134.16	138.56	142.83
	11	0	33.166	46.904	57.446	66.332	74.162	81.24	87.75	93.808	99.499	104.88	110	114.89	119.58	124.1	128.45	132.66	136.75
	10	0	31.623	44.721	54.772	63.246	70.711	77.46	83.666	89.443	94.868	100	104.88	109.54	114.02	118.32	122.47	126.49	130.38
	9	0	30	42.426	51.962	60	67.082	73.485	79.373	84.853	90	94.868	99.499	103.92	108.17	112.25	116.19	120	123.69
	8	0	28.284	40	48.99	56.569	63.246	69.282	74.833	80	84.853	89.443	93.808	97.98	101.98	105.83	109.54	113.14	116.62
	7	0	26.458	37.417	45.826	52.915	59.161	64.807	70	74.833	79.373	83.666	87.75	91.652	95.394	98.995	102.47	105.83	109.09
	6	0	24.495	34.641	42.426	48.99	54.772	60	64.807	69.282	73.485	77.46	81.24	84.853	88.318	91.652	94.868	97.98	101
	5	0	22.361	31.623	38.73	44.721	50	54.772	59.161	63.246	67.082	70.711	74.162	77.46	80.623	83.666	86.603	89.443	92.195
	4	0	20	28.284	34.641	40	44.721	48.99	52.915	56.569	60	63.246	66.332	69.282	72.111	74.833	77.46	80	82.462
	3	0	17.321	24.495	30	34.641	38.73	42.426	45.826	48.99	51.962	54.772	57.446	60	62.45	64.807	67.082	69.282	71.414
	2	0	14.142	20	24.495	28.284	31.623	34.641	37.417	40	42.426	44.721	46.904	48.99	50.99	52.915	54.772	56.569	58.31
	1	0	10	14.142	17.321	20	22.361	24.495	26.458	28.284	30	31.623	33.166	34.641	36.056	37.417	38.73	40	41.231
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**

Figure 3, $Q=10L^{.5}K^{.5}$

Labour

The production matrix can be perceived as a long-run model. The technology that relates to the production process is embedded in the values of A, $\propto$ and β. Each product and each technology are represented by different values of A, $\propto$ and β. A change in any of these values will alter the output values in each cell of the matrix.

1. In the production matrix that represents the production function $Q_X = 10\ L^{.5}\ K^{.5}$, there are "constant returns to scale." The exponents $\propto + \beta = 1$. Constant returns is the condition where a given percentage increase in the inputs results in the same percentage increase in output. For example in Figure 3 if L=1 and K=1, the output is 10. If both L and K are doubled (L=2, K=2) the output will double from 10 to 20 units. If both inputs are increased by 400% (from L=2, K=2) to L=10 and K= 10, output will increase by 400% from 20 to 100 units.

   Another example: If L=2 and K=4 the Q=28.294. If L and K increase by 100% (L'=4 and K'=8), the output (Q') will increase 100% to 56.569 units (there are some rounding errors).

2. Consider a change in technology or a different product that has the production function:

$$Q_X = 5\,L^{.6}\,K^{.3}$$

KAPITAL	K																		
K	**20**	0	12.282	18.616	23.744	28.217	32.26	35.989	39.476	42.769	45.901	48.897	51.774	54.549	57.233	59.835	62.364	64.826	67.228
A	**19**	0	12.095	18.332	23.381	27.786	31.767	35.439	38.874	42.116	45.2	48.15	50.984	53.716	56.359	58.921	61.412	63.836	66.201
P	**18**	0	11.9	18.037	23.005	27.339	31.256	34.869	38.248	41.439	44.473	47.375	50.163	52.852	55.452	57.973	60.424	62.809	65.136
I	**17**	0	11.698	17.731	22.614	26.875	30.725	34.276	37.598	40.734	43.717	46.57	49.311	51.953	54.509	56.988	59.396	61.741	64.029
T	**16**	0	11.487	17.411	22.206	26.39	30.171	33.659	36.92	40	42.929	45.731	48.422	51.017	53.527	55.961	58.326	60.629	62.875
A	**15**	0	11.267	17.077	21.781	25.884	29.592	33.013	36.212	39.233	42.106	44.854	47.493	50.039	52.5	54.888	57.207	59.466	61.669
L	**14**	0	11.036	16.727	21.334	25.354	28.986	32.337	35.47	38.429	41.243	43.935	46.52	49.014	51.425	53.763	56.035	58.248	60.406
	13	0	10.793	16.36	20.865	24.796	28.349	31.626	34.691	37.584	40.337	42.969	45.498	47.936	50.294	52.581	54.803	56.967	59.078
	12	0	10.537	15.971	20.37	24.208	27.676	30.876	33.868	36.693	39.379	41.949	44.418	46.799	49.101	51.334	53.503	55.616	57.676
	11	0	10.266	15.56	19.845	23.584	26.963	30.08	32.995	35.747	38.365	40.868	43.274	45.593	47.836	50.011	52.125	54.183	56.19
	10	0	9.9763	15.121	19.286	22.92	26.203	29.232	32.065	34.74	37.283	39.716	42.054	44.308	46.487	48.601	50.655	52.655	54.606
	9	0	9.6659	14.651	18.686	22.206	25.388	28.323	31.067	33.659	36.123	38.481	40.745	42.929	45.041	47.089	49.079	51.017	52.907
	8	0	9.3303	14.142	18.037	21.435	24.506	27.339	29.989	32.49	34.869	37.145	39.331	41.439	43.477	45.454	47.375	49.246	51.07
	7	0	8.9639	13.587	17.329	20.594	23.544	26.266	28.811	31.214	33.5	35.686	37.786	39.811	41.77	43.669	45.515	47.312	49.065
	6	0	8.5588	12.973	16.546	19.663	22.48	25.079	27.509	29.804	31.986	34.073	36.079	38.012	39.882	41.696	43.458	45.174	46.847
	5	0	8.1033	12.282	15.665	18.616	21.283	23.744	26.045	28.217	30.284	32.26	34.158	35.989	37.76	39.476	41.145	42.769	44.354
	4	0	7.5786	11.487	14.651	17.411	19.905	22.206	24.358	26.39	28.323	30.171	31.947	33.659	35.315	36.92	38.481	40	41.482
	3	0	6.9519	10.537	13.439	15.971	18.259	20.37	22.344	24.208	25.981	27.676	29.305	30.876	32.395	33.868	35.299	36.693	38.052
	2	0	6.1557	9.3303	11.9	14.142	16.168	18.037	19.785	21.435	23.005	24.506	25.949	27.339	28.684	29.989	31.256	32.49	33.694
	1	0	5	7.5786	9.6659	11.487	13.133	14.651	16.07	17.411	18.686	19.905	21.077	22.206	23.299	24.358	25.388	26.39	27.368
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**
														Labour					

Figure 4, $Q=5L^{.6}K^{.3}$

Decreasing Returns

The production matrix for the new production function is shown in Figure 4. Since $\propto + \beta = .9$, the new production function demonstrates "decreasing returns to scale." A given percentage increase in both inputs will result in a smaller increase in output.

Note that if L=2 and K=2, Q=9.33. If L and K increase 100% (L=4, K=4) output increases by less than 100% from 9.33 to 17.4. This is decreasing returns to scale.

3. Increasing returns to scale is associated with a production function where $\propto + \beta > 1$. In Figure 5 a production function with increasing returns is shown,

$$Q_X = 3L^{.6}K^{.7}$$

If L=4 and K=2, Q= 11.196. If both L and K are doubled, output will more than double from 11.196 to 26.39.

K	**20**	0	24.425	37.022	47.219	56.115	64.154	71.57	78.506	85.054	91.283	97.239	102.96	108.48	113.82	118.99	124.02	128.92	133.69
A	**19**	0	23.564	35.716	45.553	54.136	61.891	69.046	75.737	82.055	88.063	93.81	99.331	104.65	109.8	114.8	119.65	124.37	128.98
P	**18**	0	22.689	34.39	43.862	52.125	59.593	66.482	72.924	79.007	84.793	90.326	95.642	100.77	105.73	110.53	115.2	119.75	124.19
I	**17**	0	21.799	33.041	42.141	50.081	57.256	63.874	70.064	75.908	81.467	86.783	91.891	96.815	101.58	106.2	110.69	115.06	119.32
T	**16**	0	20.893	31.668	40.39	48	54.877	61.22	67.153	72.754	78.082	83.177	88.073	92.793	97.358	101.78	106.09	110.28	114.36
A	**15**	0	19.97	30.269	38.606	45.88	52.453	58.516	64.187	69.541	74.633	79.503	84.182	88.694	93.057	97.289	101.4	105.4	109.31
L	**14**	0	19.029	28.842	36.786	43.717	49.98	55.757	61.16	66.262	71.114	75.755	80.213	84.512	88.67	92.702	96.62	100.43	104.16
	13	0	18.067	27.384	34.926	41.507	47.453	52.939	58.068	62.912	67.519	71.925	76.158	80.24	84.187	88.015	91.735	95.357	98.89
	12	0	17.082	25.892	33.023	39.245	44.867	50.054	54.904	59.484	63.84	68.006	72.008	75.868	79.6	83.219	86.737	90.161	93.501
	11	0	16.073	24.362	31.072	36.926	42.216	47.096	51.66	55.969	60.068	63.988	67.754	71.385	74.897	78.302	81.611	84.834	87.976
	10	0	15.036	22.79	29.067	34.543	39.491	44.057	48.326	52.357	56.191	59.858	63.381	66.777	70.063	73.248	76.344	79.358	82.298
	9	0	13.967	21.169	27	32.087	36.684	40.924	44.89	48.635	52.196	55.602	58.874	62.03	65.081	68.041	70.916	73.716	76.447
	8	0	12.861	19.494	24.863	29.547	33.781	37.686	41.337	44.786	48.065	51.202	54.215	57.121	59.931	62.656	65.304	67.882	70.397
	7	0	11.714	17.754	22.644	26.911	30.766	34.323	37.649	40.789	43.776	46.633	49.377	52.023	54.583	57.065	59.476	61.825	64.115
	6	0	10.515	15.938	20.328	24.158	27.619	30.812	33.798	36.617	39.298	41.863	44.326	46.702	49	51.228	53.393	55.501	57.557
	5	0	9.2555	14.029	17.893	21.264	24.31	27.12	29.748	32.23	34.59	36.847	39.015	41.106	43.129	45.09	46.995	48.851	50.661
	4	0	7.917	12	15.305	18.189	20.794	23.198	25.446	27.569	29.588	31.518	33.373	35.162	36.892	38.569	40.199	41.786	43.334
	3	0	6.473	9.8112	12.514	14.871	17.002	18.967	20.805	22.54	24.191	25.77	27.286	28.748	30.163	31.534	32.867	34.165	35.43
	2	0	4.8735	7.3869	9.4214	11.196	12.8	14.28	15.664	16.971	18.213	19.402	20.544	21.645	22.71	23.742	24.746	25.723	26.675
	1	0	3	4.5471	5.7995	6.8922	7.8796	8.7905	9.6423	10.447	11.212	11.943	12.646	13.324	13.979	14.615	15.233	15.834	16.421
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**

Labour

Figure 5, $Q=3L^{.6}K^{.7}$

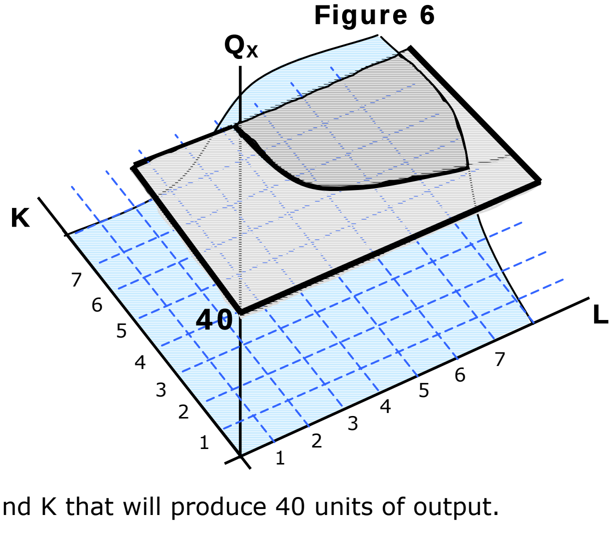
Increasing Returns

D. More Complicated Production Functions may result in a condition where there are increasing returns to scale initially then constant and decreasing returns to scale set in as the size of the production process and output increase. The phrases "economies of scale" and "economies of mass production" are associated with increasing returns to scale. The concept of returns to scale requires technology to remain constant and the inputs to be used in constant proportions. The general statement "economies of scale" is less precise and can mean a variety of things. The least specific version is that as output increases, the long-run cost per unit (long-run average cost) falls. If the K/L ratio is not constant the LRAC may fall because there is some "unused inputs" at lower levels of production. This may be because the inputs are "lumpy" or in discrete or indivisible units. The cost per unit might fall because the price of inputs has changed or there has been a change the prices of the inputs (This wouldn't really be a fall in LRAC because the *ceteris paribus* conditions has been violated).

E. The Isoquant

A given amount of output (Q_X) can be produced with different combinations of labour and kapital. All combinations of labour and kapital that produce the same quantity of output is called and "isoquant." In Figure 6, a production surface is shown. A plane is passed horizontally through the surface at an elevation of 40 units of output. Where the plane "cuts" the surface an intersection or contour line is formed. This contour line or "isoquant" shows all combinations of L and K that will produce 40 units of output.

Figure 6



The isoquant can also be shown on a production matrix. In Figure 7 each cell that has an output of 40 units is shaded in red. Columns where there is no cell with a value of 40 shown would have a cell with a value 40 if L and K were divided into smaller units of input.

KAPITAL	K	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
K	20	0	44.721	63.246	77.46	89.443	100	109.54	118.32	126.49	134.16	141.42	148.32	154.92	161.25	167.33	173.21	178.89	184.39
A	19	0	43.589	61.644	75.498	87.178	97.468	106.77	115.33	123.29	130.77	137.84	144.57	151	157.16	163.1	168.82	174.36	179.72
P	18	0	42.426	60	73.485	84.853	94.868	103.92	112.25	120	127.28	134.16	140.71	146.97	152.97	158.75	164.32	169.71	174.93
I	17	0	41.231	58.31	71.414	82.462	92.195	101	109.09	116.62	123.69	130.38	136.75	142.83	148.66	154.27	159.69	164.92	170
T	16	0	40	56.569	69.282	80	89.443	97.98	105.83	113.14	120	126.49	132.66	138.56	144.22	149.67	154.92	160	164.92
A	15	0	38.73	54.772	67.082	77.46	86.603	94.868	102.47	109.54	116.19	122.47	128.45	134.16	139.64	144.91	150	154.92	159.69
L	14	0	37.417	52.915	64.807	74.833	83.666	91.652	98.995	105.83	112.25	118.32	124.1	129.61	134.91	140	144.91	149.67	154.27
	13	0	36.056	50.99	62.45	72.111	80.623	88.318	95.394	101.98	108.17	114.02	119.58	124.9	130	134.91	139.64	144.22	148.66
	12	0	34.641	48.99	60	69.282	77.46	84.853	91.652	97.98	103.92	109.54	114.89	120	124.9	129.61	134.16	138.56	142.83
	11	0	33.166	46.904	57.446	66.332	74.162	81.24	87.75	93.808	99.499	104.88	110	114.89	119.58	124.1	128.45	132.66	136.75
	10	0	31.623	44.721	54.772	63.246	70.711	77.46	83.666	89.443	94.868	100	104.88	109.54	114.02	118.32	122.47	126.49	130.38
	9	0	30	42.426	51.962	60	67.082	73.485	79.373	84.853	90	94.868	99.499	103.92	108.17	112.25	116.19	120	123.69
	8	0	28.284	40	48.99	56.569	63.246	69.282	74.833	80	84.853	89.443	93.808	97.98	101.98	105.83	109.54	113.14	116.62
	7	0	26.458	37.417	45.826	52.915	59.161	64.807	70	74.833	79.373	83.666	87.75	91.652	95.394	98.995	102.47	105.83	109.09
	6	0	24.495	34.641	42.426	48.99	54.772	60	64.807	69.282	73.485	77.46	81.24	84.853	88.318	91.652	94.868	97.98	101
	5	0	22.361	31.623	40	44.721	50	54.772	59.161	63.246	67.082	70.711	74.162	77.46	80.623	83.666	86.603	89.443	92.195
	4	0	20	28.284	34.641	40	44.721	48.99	52.915	56.569	60	63.246	66.332	69.282	72.111	74.833	77.46	80	82.462
	3	0	17.321	24.495	30	34.641	40	42.426	45.826	48.99	51.962	54.772	57.446	60	62.45	64.807	67.082	69.282	71.414
	2	0	14.142	20	24.495	28.284	31.623	40	40	40	42.426	44.721	46.904	48.99	50.99	52.915	54.772	56.569	58.31
	1	0	10	14.142	17.321	20	22.361	24.495	26.458	28.284	40	40	33.166	34.641	36.056	37.417	38.73	40	41.231
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

Figure 7, $Q=10L^{.5}K^{.5}$

Labour

Connecting each of the shaded red cells will yield an isoquant for 40 units of production.

F. Characteristics of Isoquants

Isoquants are similar to indifference curves and have many of the same characteristics. While indifference curves measure utility or satisfaction as a function of two goods, isoquants measure output as a function of two inputs. The slope (MRS or marginal rate of substitution) of the indifference curve measures the rate at which an individual is "willing" to exchange one good for another holding the level of utility constant. The slope of the isoquant is called the "marginal rate of technical substitution" (MRTS).

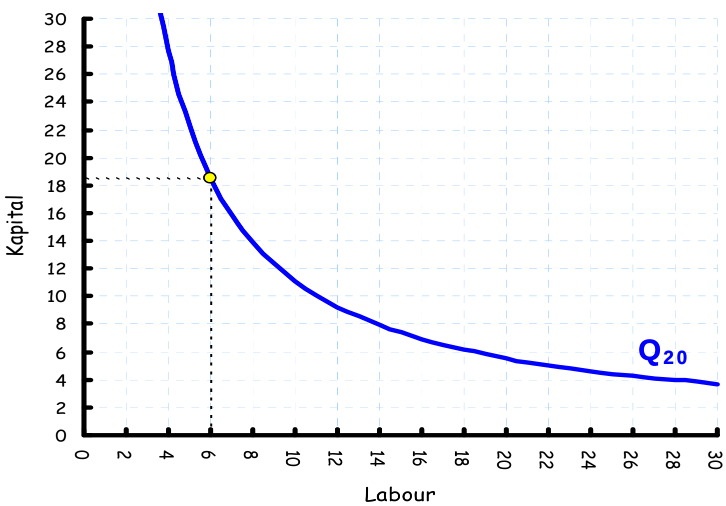
Normally, isoquants are:

- Everywhere dense, there is one isoquant for every possible level of output. If output is continuous there are an infinite number of isoquants
- Negatively sloped
- Convex sets
- They never touch or intersect (you cant have two different levels of output at the same time)
- For every production function there is a different isoquant map that has an infinite number of isoquants

An increase in the values of A, $\propto$, β will "shift" each isoquant in toward the point of origin, i.e. it takes fewer resources to produce the same level of output.

In Figure 8 an isoquant for 20 units of output from the production process $Q = 1.9L^{.5}K^{.5}$ is shown.

Figure 8



In Figure 8 the production function is $Q = 1.9L^{.5}K^{.5}$. All combinations of L and K that will produce 20 units of output fall on the Isoquant, Q_{20}.

If 6 units of labour are used in the production process, 18.5 units of K are required to produce 20 units of output.

If L is increased to 8 units, 13.9 units of K will be required to produce 20 units of output. It is possible to substitute 2 units of labour (8-6=2) for 4.6 units of K and still produce 20 units of output.

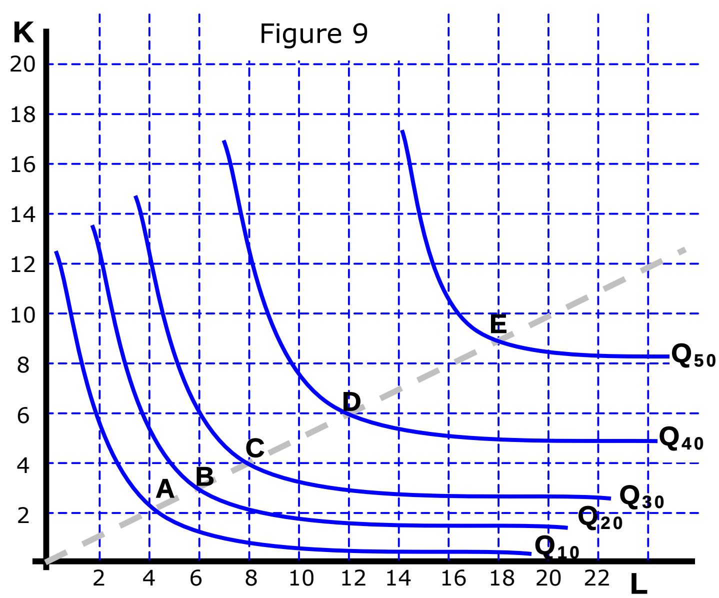
The slope of the isoquant represents the rate at which L can be substituted for K without altering the level of output (Q). The absolute value of the slope of the isoquant ($\Delta K/\Delta L$) is the MRS.

VII. Increasing, Constant and Decreasing Returns to Scale

In a more complicated production function it may be possible to have increasing returns to scale as a production process begins to expand. After all the gains from increasing returns are exhausted, constant returns may set in. Eventually, a production process may reach decreasing returns to scale. These three conditions will be shown on an isoquant map by the "spacing" of unit isoquants. In Figure 9 the pattern is shown.

Table 9
Spacing of Isoquants and Returns

L	K	%Δ Inputs	Q	%ΔQ	Point
4	2		12		A
6	3	50%	20	100%	B
8	4	33%	30	50%	C
12	6	50%	40	33%	D
18	9	50%	50	25%	E



In Figure 9 and Table 9, the spacing of the isoquants shows increasing and decreasing returns to scale. As production is increased from Q=10 at point A where L=4 and K=2 to Q=20 at point B, the inputs increase by 50% while the output increases by 100%. This is increasing returns to scale. Note that the K/L ratio remains constant at .5. As the inputs are increased from L=4, K=2 to L'=6, K'=3, the inputs have increased by 33% while output increases by 50%. Still increasing returns.

To increase output from Q'=30 to Q"=40 (an increase of 33%) the inputs increase from L'=6, K'=3 to L"=12, K"=6 (an increase of 50%). This is decreasing returns to scale.

Note: in Table and Figure 9 the input combinations A, B,…E are not necessarily the least cost per unit. They are selected because they are on a ray from the point of origin which means they all have the same K/L ratio. Least cost/unit combinations of inputs is determined by:

$$MRTS = \left| -\frac{P_L}{P_K} \right|$$ This is analgous to the utility maximization with indifference curves.

VIII. Short-Run Production Function (Slice and Dice The Production Matrix)

In the short-run, at least one input is fixed and one is variable. (Technology is constant.) The fixed input is frequently used to give information about the size of a plant.

If the long-run production function is of the form Q = f(L, K) *ceteris paribus.* If K is held constant the short-run production takes the form Q = f(L) where K is fixed. It appears as a row in the production matrix. If labour is fixed, the short-run production function would be of the form Q = f(K) given L and technology.

Using the production function $Q = 10L^{.5}K^{.5}$ and K is given as 16, the short-run production function will be $Q = (10)L^{.5}(16)^{.5} = (40)L^{.5}$.

K	20	0	44.721	63.246	77.46	89.443	100	109.54	118.32	126.49	134.16	141.42	148.32	154.92	161.25	167.33	173.21	178.89	184.39
A	19	0	43.589	61.644	75.498	87.178	97.468	106.77	115.33	123.29	130.77	137.84	144.57	151	157.16	163.1	168.82	174.36	179.72
P	18	0	42.426	60	73.485	84.853	94.868	103.92	112.25	120	127.28	134.16	140.71	146.97	152.97	158.75	164.32	169.71	174.93
I	17	0	41.231	58.31	71.414	82.462	92.195	101	109.09	116.62	123.69	130.38	136.75	142.83	148.66	154.27	159.69	164.92	170
T	16	0	40	56.569	69.282	80	89.443	97.98	105.83	113.14	120	126.49	132.66	138.56	144.22	149.67	154.92	160	164.92
A	15	0	38.73	54.772	67.082	77.46	86.603	94.868	102.47	109.54	116.19	122.47	128.45	134.16	139.64	144.91	150	154.92	159.69
L	14	0	37.417	52.915	64.807	74.833	83.666	91.652	98.995	105.83	112.25	118.32	124.1	129.61	134.91	140	144.91	149.67	154.27
	13	0	36.056	50.99	62.45	72.111	80.623	88.318	95.394	101.98	108.17	114.02	119.58	124.9	130	134.91	139.64	144.22	148.66
	12	0	34.641	48.99	60	69.282	77.46	84.853	91.652	97.98	103.92	109.54	114.89	120	124.9	129.61	134.16	138.56	142.83
	11	0	33.166	46.904	57.446	66.332	74.162	81.24	87.75	93.808	99.499	104.88	110	114.89	119.58	124.1	128.45	132.66	136.75
	10	0	31.623	44.721	54.772	63.246	70.711	77.46	83.666	89.443	94.868	100	104.88	109.54	114.02	118.32	122.47	126.49	130.38
	9	0	30	42.426	51.962	60	67.082	73.485	79.373	84.853	90	94.868	99.499	103.92	108.17	112.25	116.19	120	123.69
	8	0	28.284	40	48.99	56.569	63.246	69.282	74.833	80	84.853	89.443	93.808	97.98	101.98	105.83	109.54	113.14	116.62
	7	0	26.458	37.417	45.826	52.915	59.161	64.807	70	74.833	79.373	83.666	87.75	91.652	95.394	98.995	102.47	105.83	109.09
	6	0	24.495	34.641	42.426	48.99	54.772	60	64.807	69.282	73.485	77.46	81.24	84.853	88.318	91.652	94.868	97.98	101
	5	0	22.361	31.623	38.73	44.721	50	54.772	59.161	63.246	67.082	70.711	74.162	77.46	80.623	83.666	86.603	89.443	92.195
	4	0	20	28.284	34.641	40	44.721	48.99	52.915	56.569	60	63.246	66.332	69.282	72.111	74.833	77.46	80	82.462
	3	0	17.321	24.495	30	34.641	38.73	42.426	45.826	48.99	51.962	54.772	57.446	60	62.45	64.807	67.082	69.282	71.414
	2	0	14.142	20	24.495	28.284	31.623	34.641	37.417	40	42.426	44.721	46.904	48.99	50.99	52.915	54.772	56.569	58.31
	1	0	10	14.142	17.321	20	22.361	24.495	26.458	28.284	30	31.623	33.166	34.641	36.056	37.417	38.73	40	41.231
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
														Labour					

Figure 10, $Q=10L^{.5}K^{.5}$

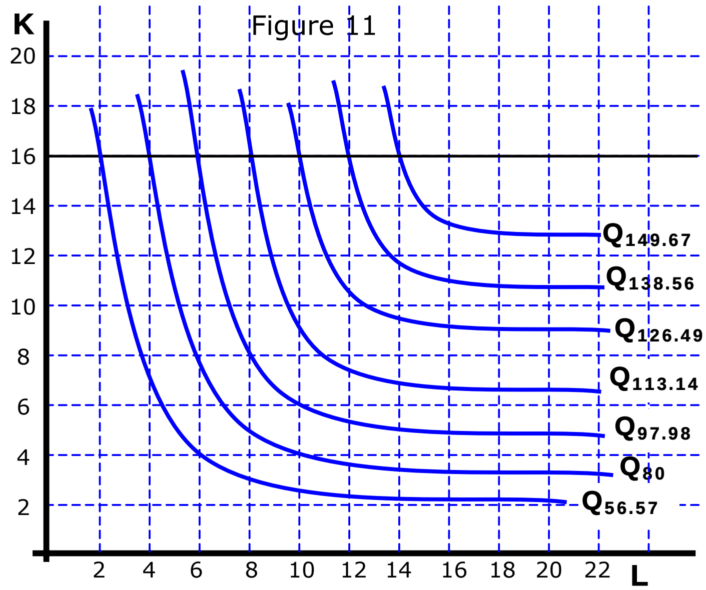
In Figure 10, this production function can be seen as row K=16. The data from row K=16 in Figure 10 is reproduced in Table 10 as a short run production function.

If labour were the fixed input the short-run production function would be a column in the production matrix.

Note that there are as many short-run production functions as there are potential fixed inputs.

The isoquant map can also be used to show the short-run production function. If Figure 11 a cross section is taken of the production surface at the amount of the fixed input. The line or row along the isoquants

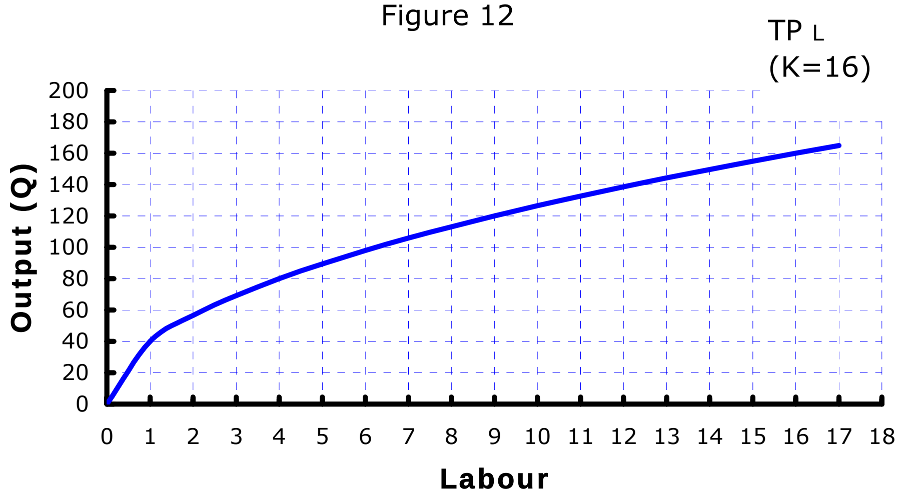
Table 10 Short-Run Production Function		
K	L	Q
16	0	0.00
16	1	40.00
16	2	56.57
16	3	69.28
16	4	80.00
16	5	89.44
16	6	97.98
16	7	105.83
16	8	113.14
16	9	120.00
16	10	126.49
16	11	132.66
16	12	138.56
16	13	144.22
16	14	149.67
16	15	154.92
16	16	160.00
16	17	164.92



where K = 16 is intersected by isoquants where labour is 2, 4, 6, 8, 10,… Each isoquant the interects the line has a quantity that is the same as in Table 10, or when $Q = 40\ L^{.5}$.

The quantities shown in Table 10 or the values of the isoquants that intersect the line where K=16 can be plotted in Table 12. This is "like" cutting the production surface along the line K=16 and looking at the cross section.

Total Product L (K=16)
Figure 12



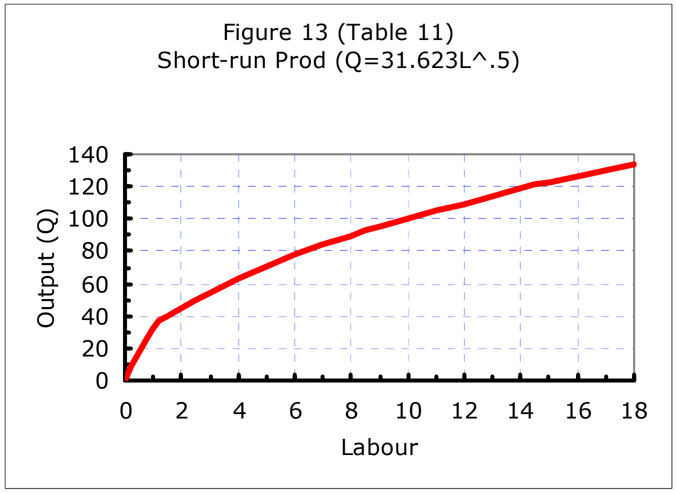
Short-run production functions can be developed for any level of fixed input or any fixed input.

A. Another level of fixed inputs

Figure 10 represents a Cobb-Douglas production function ($Q=10L^{.5}K^{.5}$). In Figure 11 a cross section is taken where K = 16. In Figure 12 the view is of the "end" of the cross section. In Figure 13, a cross section is taken where K=10. Table 11 is the row where K=10 in the production map shown in Figure 10.

Table 11
Cobb-Douglas
($Q=10L^{.5}K^{.5}$)
Short-Run Production Function Q=31.623L.5
(Where K = 10)

K = 10	Labour	Q = TP_L
10	0	0.00
10	1	31.62
10	2	44.72
10	3	54.77
10	4	63.25
10	5	70.71
10	6	77.46
10	7	83.67
10	8	89.44
10	9	94.87
10	10	100.00
10	11	104.88
10	12	109.55
10	13	114.02
10	14	118.32
10	15	122.48
10	16	126.49
10	17	130.38
10	18	134.17



If labour were the fixed input, the cross section would be a column at that quantity of labour. Note that the short-run production function is determined not only by the technology embedded in the long-run production function but also the level of the fixed input. For each long-run production function there are an infinite number of possible short-run production functions.

IX. A Production Function That Demonstrates Increasing, Constant and Decreasing Returns to Scale

While a Cobb-Douglass production function demonstates only increasing, decreasing or constant returns to scale, many production processes intially have increasing returns until constant returns are reached. If the production process continues to expand, decreasing returns may set in.

A. Increasing returns to scale are sometimes referred to as "economies of scale." Bela Gold et.al. point out that if "economies of scale" is defined precisely, it will probably never occur. If defined loosely, it is so general it is almost meaningless. Increasing returns requires that the K/L ratio remain constant. Increasing returns tends to be the result of division of labour. Economies of scale (loosely defined) is due to the use of previously unused capacity.

When increasing returns to scale exist, the cross section at a level of fixed input will be increasing at an increasing rate. The cross section is the "total product of the variable factor." If labour is variable and kapital is fixed the function shows Q as a function of L given K, (Q = f(L), cet. par.

The first derivative of the TP_L function is the "marginal product of labour" (MP_L). If you hold labour constant and let $TP_K = f(K)$ given L, the first derivative of TP_K is the marginal product of K (MP_K).

B. The short-run production function will take the form

$Q = f(L)$,

for a given technology and plant size. A change in any of the fixed inputs or technology will alter the short-run production function.

Total product (**TP** or **Q**) is the total output. Q or TP = f(L) given a fixed size of plant and technology.

Average product (AP) is the output per unit of input. AP = TP/L (in this case the output per worker). **AP_L** is the average product of labour.

C. **Marginal Product** (**MP_L**) is the change in output "caused" by a change in the variable input (L), $MP_L = \frac{\Delta TP}{\Delta L}$

D. Total and Marginal Product

Over the range of inputs there are four possible relationships between Q and L:

- TP or Q can increase at an increasing rate. MP will increase, (In the figure below, this range is from O to L_A.)
- TP may pass through an inflection point, in which case MP will be a maximum. (In the figure below Figure 1.pc, this is point A at L_A amount of input.) TP or Q may increase at a constant rate over some range of output. In this case, MP will remain constant in this range.
- TP might increase at a decreasing rate. This will cause MP to fall. This is referred to as "diminishing MP." In Figure 1.pc, this is shown in the range from L_A to L_B.

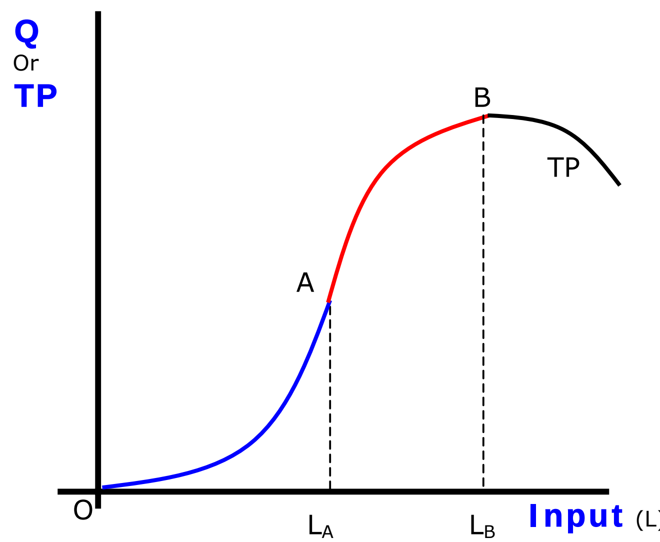


Figure 1.pc

- If "too many" units of the variable input are added to the fixed input, TP can decrease, MP will be negative. Any addition of L beyond LB will reduce output, the MP of the input will be negative. It would be foolish to continue adding an input (even if it were "free") when the MP is negative.

The relationship between the total product (TP) and the marginal product (MP) can be shown. In the figure to the right (Figure 2.pc), note that the inflection point in the TP function is at the same level of input (L_A) as the maximum of the MP. It is also important to understand that the maximum of the TP occurs when the MP of the input is zero at L_B.

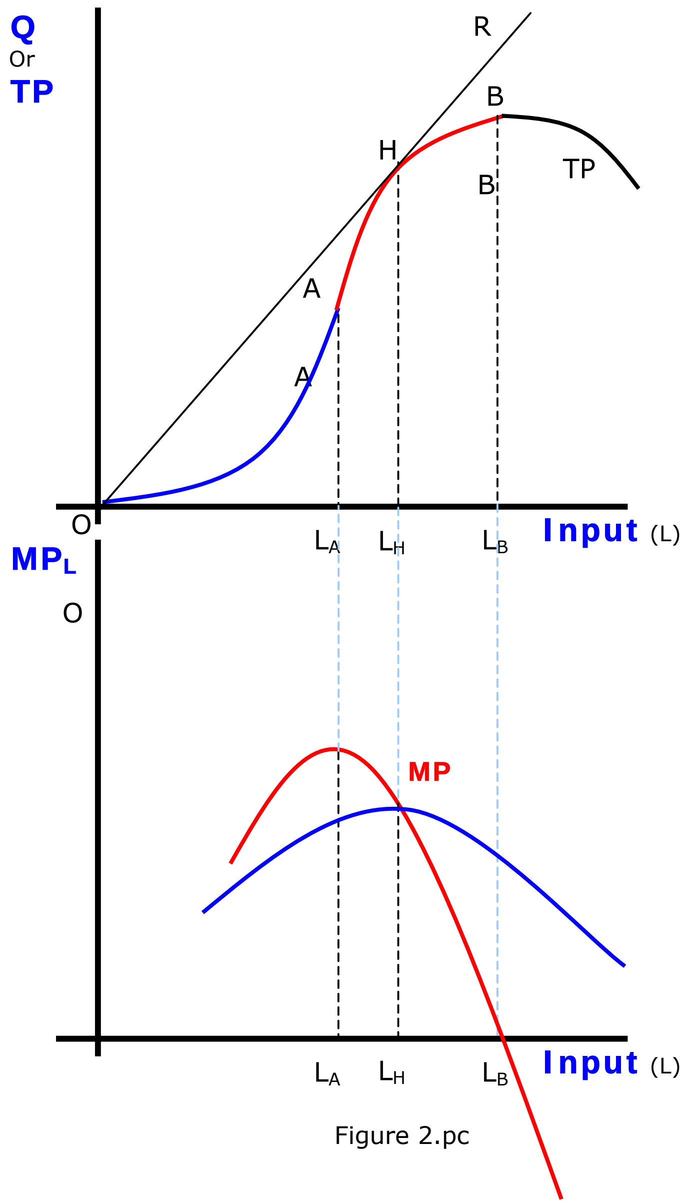


Figure 2.pc

Average, Marginal and Total Product

The average product (AP) is related to both the TP and MP. Construct a ray (OR in Figure 2.pc) from the origin to a tangent point (H) on the TP. This will locate the level of input where the AP is a maximum, L_H. Note that at L_H level of input, AP_L is a maximum and is equal to the MP_L. When the MP is greater than the AP, MP "pulls" AP up. When MP is less than AP, it "pulls" AP down. MP will always intersect the AP at the maximum of the AP.

Technical efficiency was defined as a ratio of output to input,

$$Technical\ Efficiency = \frac{Output}{Input}$$

The AP is a ratio of TP or Q or output to a variable input and a set of fixed input(s).

$$AP = \frac{TP}{L} = \frac{output\ (Q)}{input\ (L+K)}$$

The maximum of the AP is the "technically efficient" use of the variable input (L) given plant size. Remember that K is fixed in the short-run.

E. Review of Production Relationships

In the short-run, as a variable input is added to a fixed input (plant size) the TP may increase at an increasing rate. As TP increases at an increasing rate MP for the variable input will rise. So long as the MP is greater than the AP of the variable input, AP will rise. This range is caused by a more "efficient mix" of inputs. Initially, there is "too much" of the fixed input and not enough of the variable input.

Eventually, as more variable inputs are added there may be an inflection point in the TP. It is also possible that the TP might increase at a constant rate in a range. An inflection point in the TP is where the "curvature" of the TP changes; it is changing from increasing at an increasing rate (concave from above or convex from below) to increasing at a decreasing rate (convex from above or concave from below). At this point, the MP of the variable input will be a maximum. AP will be rising.

At some point, the TP will begin to increase at a decreasing rate. There is "too much variable input" for the fixed input. Productivity of each additional input will be less, MP will fall in this range. AP of the variable input may be greater or less than the MP in this range. If MP is greater than AP, AP will be increasing. If MP is less than AP, AP will be decreasing.

A ray from the origin and tangent to the TP function (OR in Figure 2.pc) will identify the level of variable input where the AP will be a maximum. At this point MP will equal AP. Since the fixed input is constant, AP is the equivalent of out measure of technical efficiency;

$$Tech.\,Efficiency = \frac{output}{input} = \frac{TP}{L + (\,fixed\ input)} = \text{AP of the varible input}$$

X. Production and Cost

If the production function and the prices of inputs are known it is possible to calculate most "explicit" cost functions. External costs and transaction costs are often "implicit" costs and are typically not included in the costsfunctions of the firm or plant.

The cost functions are categorized by time periods that are consistent with the time periods used in production. Short-run costs refelect the variable inputs in the production process. The short-run costs are determined by the production function, the fixed input (or scale of plant) and the prices of the variable input(s). Fixed costs or "overhead" costs are determined by the amount and prices of the fixed input(s).

Economics, production management and industrial engineers often try to use production functions to estimate cost functions. Accountants, economists, production managers and industrial engineers may use cost data to deduce information about production functions to ascertain the effects of changes in inputs.

The cost functions are determined by the production function (technology) and the prices of the inputs. In Figure 14 the relationships between TP, AP, MP and the average variable cost (AVC) and marginal cost (MC) are shown.

Figure 14

The average variable cost (AVC) and the average product (AP) are closely related. Marginal cost (MC) reflects the marginal product (MP). In the three panels of Figure V.6, the total product (TP in the upper panel) is related to the AP and MP in the middle panel. The lower panel shows the relationship of average variable cost (AVC) and marginal cost (MC) to AP, MP and TP.

As the variable input (L in this example) increases to L_A the TP in the upper panel increases at an increasing rate to point A. In this range the marginal product (in the middle panel) will rise. When more than L_A amount of the labour input is used, the MP will decrease for each additional unit. The inflection point at A in the upper panel is consistent with the maximum of the MP at point A* in the middle panel.

Point H on the TP function (in the upper panel) is constructed by passing a ray from the origin to a point of tangency on the TP function. This identifies L_H amount of labour. In the middle panel the AP of labour will rise up to L_H amount of input. Notice that in this range the MP is above or greater than the AP. When MP>AP, the AP will be increasing. At point H* in the middle panel, AP will be a maximum. At this point MP = AP.

As the input is increased above L_H, the AP will fall. When MP < AP, AP will be decreasing. At the maximum of the TP (at point B in the upper panel) L_B amount of the variable input is used. At this level of labour (L_B), the MP will be zero. It is important to note that when MP = 0, TP is a maximum.

In the lower panel, MC will be at a minimum at the output level (Q_A) where MP is a maximum (Q_A output at L_A input). AVC will be a minimum at Q_H output. This is where MC=AVC. This is at output level Q_H produced by L_H labour. When AVC is a minimum, AP will be a maximum.

When MC<AVC, AVC will be decreasing. When MC>AVC, AVC will increase. When AVC=MC, AVC is a minimum

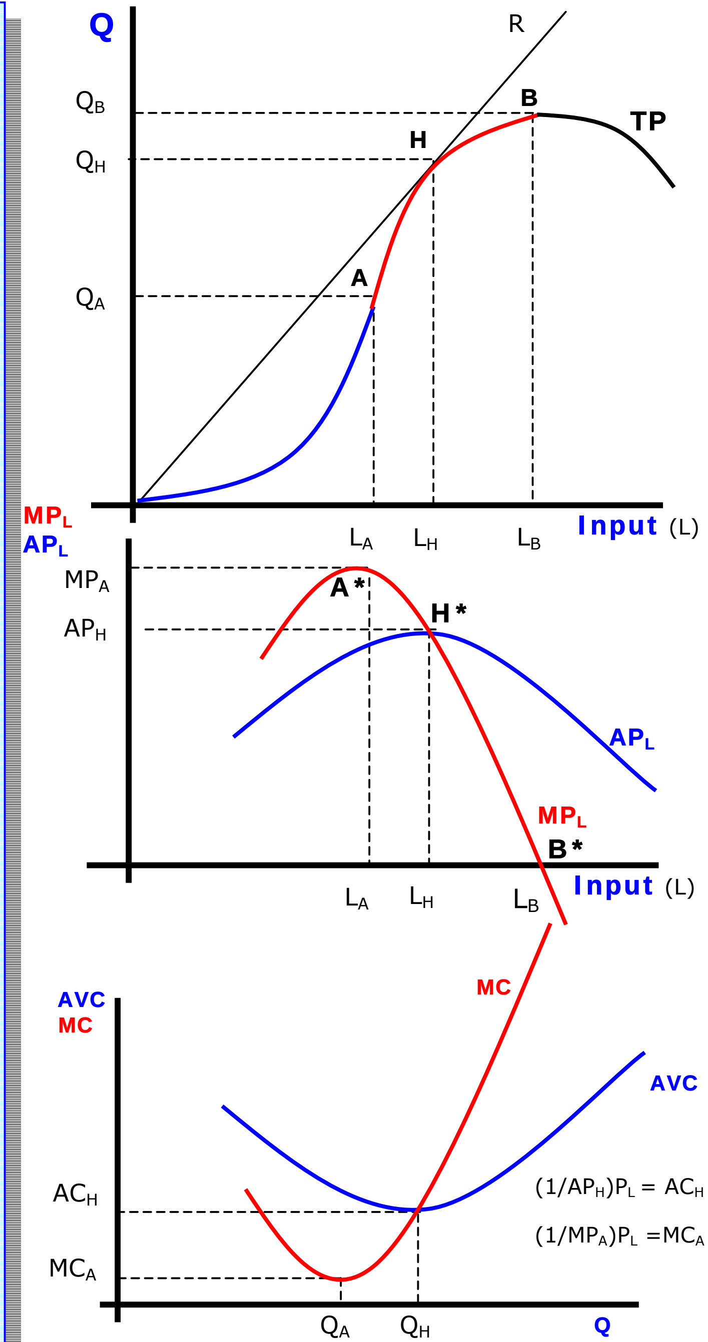


Figure 14