Calculus Review for Microeconomics

R. Larry Reynolds

Introduction

In the last half of the 18th century Isaac Newton [1642-1727] and G.W. Leibniz [1646-1716] lay the foundations for what was to become known as "calculus." This was a powerful tool to deal with infinitesimal problems of arc lengths, tangents to curves, area and volumes. The Bernoulli brothers, Jackob [1654-1705] and Johann [1667-1748] extended the tool to deal with the processes of change. During the mid 19th century (from about 1825 to 1875) a growing number of writers applied calculus to the problems of changes in the realm of economics. Johann Heinrich von Thünen [1783-1850], Hermann Heinrich Gossen [1810-1858], Jules Dupuit [1804-1866] and Augustin Cournot [1801-1877] applied mathematical tools to economic problems. In the 1870's W.S. Jevons [1835-1882], Carl Menger [1840-1921] and Léon Walras [1834-1910] initiated what has come to be known as the "Marginalist Revolution" in economics. The Marginalist Revolution established the importance of calculus in economic analysis. During the 19th and 20th centuries the role of mathematics in economics has expanded.

Functions

Models of economic relationships are abstractions that explain many of the forces that shape economic behavior. Models can be in a narrative or mathematical format. In a mathematical format, the variables may be quantified or measured. The relationships between economic variables are expressed as "functional relationships." The cost of producing a unit of output is influenced by the prices of inputs, technology, regulations and amount of output. In the shorthand of functional notations, $C_X = f_C(P_R, Q_X,$ regulations, technology,…) This is read; "the cost of a unit of X is determined by the prices of the inputs or resources (P_R), the quantity of good X produced (Q_X), the regulations, the technology and other things not shown (…). Some of these variables are easily expressed in quantitative terms. In other cases, such as "regulations" and "technology," quantification is more difficult.

A function is defined as a "rule" that associates each element of a nonempty set (called a domain) with a unique element in another nonempty set (called a co domain). A function can also be defined as a relationship between two (or more) sets, where for every value of each element in an independent set, there is one and only one value in the dependent set.

Examples of functional relationships;

Price (P) is a function of the quantity (Q), **$P = f(Q)$**, or for a specific case. **$P = 100 - 5Q$**, for every value of Q there is one value of P.

Output (Q_X) is a function of the inputs labour (L) and capital (K), **$Q_X = f(L, K, \ldots)$** for a more specific case, **$Q_X = 5.3\ L^{.6}\ K^{.3}$** , for every value of L and K there is one value of Q

A function can be graphed. When there are 2 variables, the graph is two-dimensional, i.e. the graph would have two axes. If there were three variables the graph would be three-dimensional or would have three axes. There are some tricks that allow us to show three dimensions on a two dimensional plane (a contour map is an example). If there are "n" variables then the model is n-dimensional space. While we typically only graph things in two dimensions (in rare exceptions 3) n-dimensional space is often used in mathematics.

Functions, Tables and Graphs

A functional relationship between variables can be expressed as an equation, a table, or a graph. The equation and graph display the relationship as "continuous" data while the table only contains data in specific rows and columns as "discrete" data. Continuous data is infinitely divisible while discrete data is not.

Consider the function, $\mathbf{Y = 2 + 1X - .05X^2}$. This is a "second degree" equation (there is one independent variable that is raised to the power of 2); it is nonlinear (only first-degree equations are linear). In Table 1 the values of the independent variable (X) and dependent variable (Y) are shown. By substituting each value of the independent variable (X) into the equation, $\mathbf{Y = 2 + 1X - .05X^2}$, the value of the dependent variable (Y) can be calculated.

Table 1

Independent variable X	Dependent Variable Y
1	2.95
2	3.8
3	4.55
4	5.2
5	5.75
6	6.2
7	6.55
8	6.8
9	6.95
10	7
11	6.95
12	6.8
13	6.55
14	6.2
15	5.75
16	5.2
17	4.55
18	3.8
19	2.95
20	2

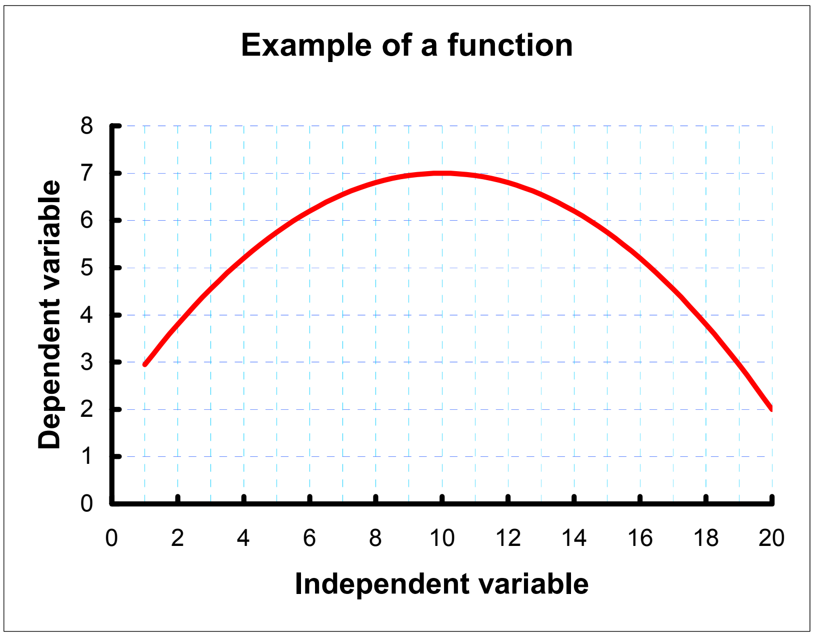


Figure 1

Slope and "Rate of Change"

The slope of a line measures the change in the dependent variable associated with a change in the independent variable. When the function is linear, the slope can be calculated as "rise over run" or $\frac{\text{Change in the dependent variable}}{\text{Change in the independent variable}} = \frac{\Delta Y}{\Delta X}$. When the function is nonlinear the slope can be imagined as the slope of an arc between two points on the function or as the slope of a tangent at a point on a function. Generally, $\frac{\Delta Y}{\Delta X}$ suggests the slope of a straight line or the slope of an arc between two points on a nonlinear function. If the change in X (ΔX) approaches the limit of 0, the notation for a derivative, $\frac{dy}{dx}$ is used. If there are several independent variables, each influencing the value of the dependent variable, the notation for a partial derivative, $\frac{\partial y}{\partial x}$ is used.

Given the function **$Q = 5.3\ L^{.6}\ K^{.3}$**, the change in Q caused by a change in L is written $\frac{\partial Q}{\partial L} = (5.3)(.6)L^{(.6-1)}K^{.3} = 3.18L^{(-.4)}K^{.3}$. This is called the "marginal product of labour." The marginal product of capital would be $\frac{\partial Q}{\partial K} = (5.3)(.3)L^{.6}K^{(.3-1)} = 1.59L^{.6}K^{(-.7)}$. The marginal product of labour (MP_L) measures the change in output (Q) associated with a change in labour. The MP_K (marginal product of capital) measures the change in Q associated with a change in K. Notice that the MP_L is influenced by the amount of K and the structure of the equation that defines the MP_K is determined by the amount of labour.

Slope of a Straight Line

Given a function **$P = 100 - 5Q$** both a table and a graph can be constructed in Table 2 and Figure 2.

Table 2

Quantity Q	Price P
1	95
2	90
3	85
4	80
5	75
6	70
7	65
8	60
9	55
10	50
11	45
12	40
13	35
14	30
15	25
16	20
17	15
18	10
19	5
20	0

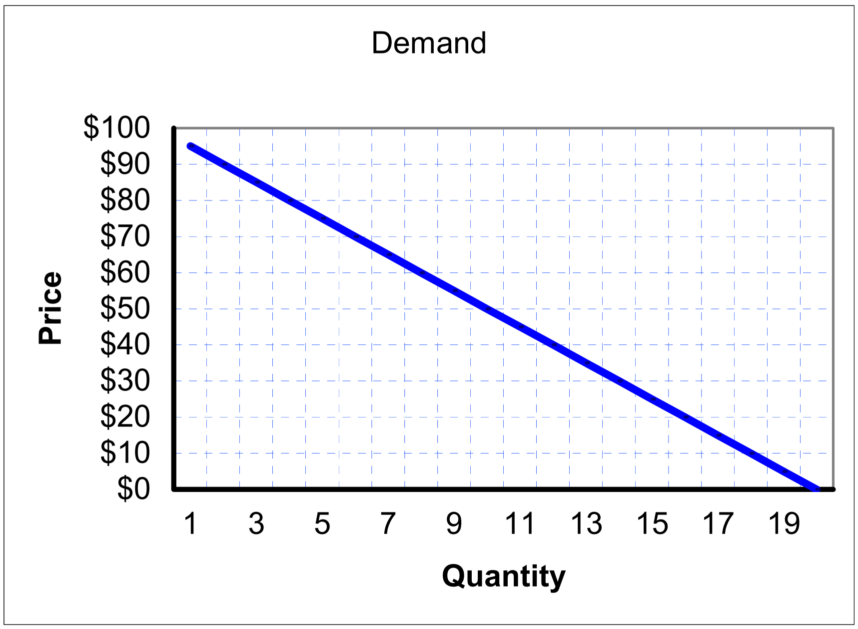


Figure 2

The slope of the line in the graph is the change in price caused by a change in quantity, for every one-unit change in Q, the value of price (P) will change by 5 units in the opposite direction. Q increases by 1, P decreases by 5, Q decreases by 3, P will increase by 15.

It may be useful to notice that the function P = 100 – 5Q can be expressed where Q = f(P). The equation would be Q = 20 - .2P. Notice that the slope of the function Q= f(P) is the reciprocal of the slope of the equivalent function P = f(Q).

Slope of a Nonlinear Function

In Figure 3 a nonlinear function (Y=f(X) is graphed. If the value of X changes from 4 to 11 then ΔX (the "run") is +7. The change in Y (ΔY or "rise") is +3. The slope of the arc between points "**a**" and "**b**" is $\frac{+3}{+7} = \frac{\text{rise}}{\text{run}} = \frac{\Delta Y}{\Delta X} = +.75$.

Notice that as the change in X (ΔX) approaches 0 the tangent **tt'**

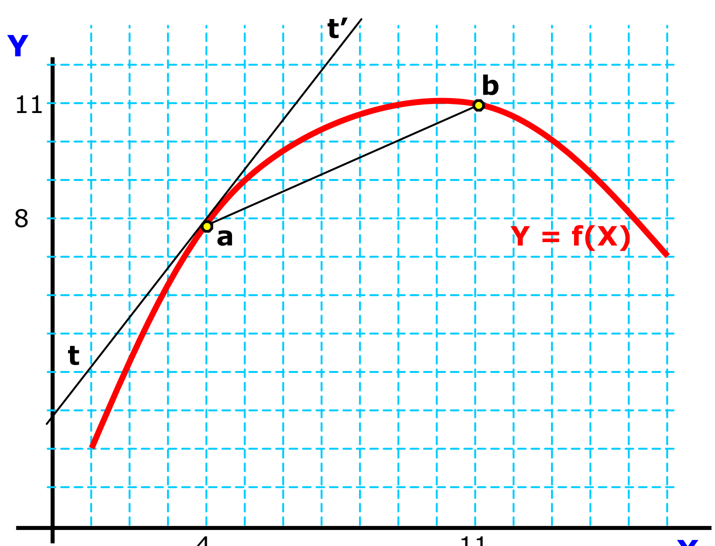


Figure 3

represents the change in Y "caused" by a change in X. The slope of **tt'** is calculated by taking the first derivative of the function, Y =f(X). If you use $\frac{\Delta y}{\Delta x}$, the slope is an estimate of the "average" slope between points a and b that lie on the function. The value of $\frac{dy}{dx}$ is the slope of the tangent at a point on the function.

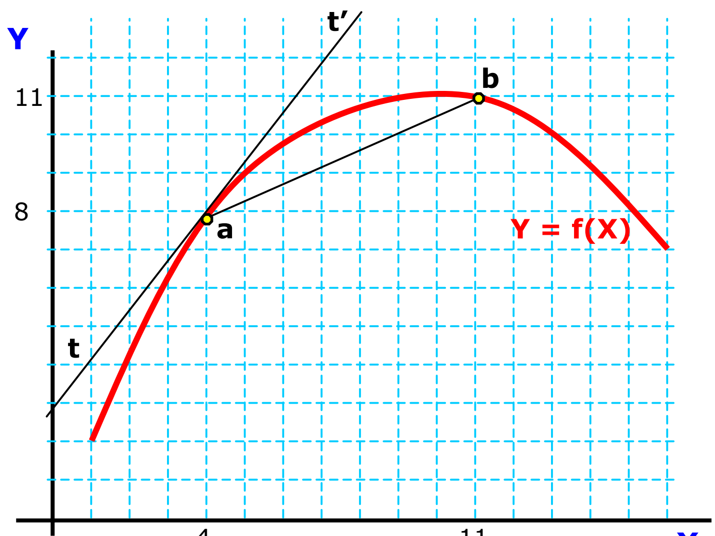


Figure 3

A Simple Rule for a Derivative

In calculus classes, much time and effort is given to proving the idea of a derivative. Here we are interested in a "quick and dirty" approach to finding the slope of the tangent at a point.

A simple rule to calculate the first derivative for a function **$Y = aX^b$** is

$$\frac{dy}{dx} = a(b)\ X^{(b-1)}.$$

This is the equation for the tangent line at any value of X.

Applying this rule to the equation, **$Y = 2 + 1X - .05X^2$** the equation can be written,

$Y = 2X^0 + 1X^1 - .05\ X^2$, this will make

$$\frac{dy}{dx} = 2(0)X^{(0-1)} + 1(1)X^{(1-1)} - .05(2)X^{(2-1)} = 1 - .1X$$

If X is 10 the value of Y is 7. [$Y = 2+1(10) - .05(10)^2 = 12 - .05(100) = 12\text{-}5 = 7$].

When X is 10 and Y is 7, the slope of the tangent to the function **$Y = 2 + 1X - .05X^2$** will be

$$\frac{dy}{dx} = 1 - .1X$$

or

$$\frac{dy}{dx} = 1 - .1X = 1 - .1(10) = 0$$

$\frac{dy}{dx}$ is the change in Y caused by a very, very, very, very small change in X ($\Delta X \rightarrow 0$).

When X = 5 (Y=5.75), the slope of the tangent to Y = f(X) is +.5.

$$\frac{dy}{dx} = 1\text{-}.1X = 1\text{-}.1(5) = +.5$$

When X=15 the slope of the tangent is -.5.

$$\frac{dy}{dx} = 1\text{-}.1X = 1\text{-}.1(15) = -.5$$

Where the slope is 0, the function has a maximum, a minimum or an inflection point. To determine which, the second derivative (or second order condition) is used.

When

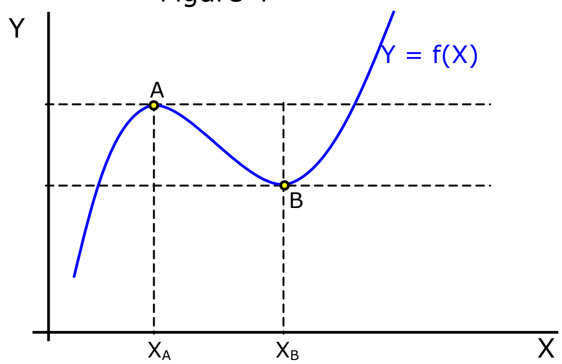
$$\frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} > 0, \text{ a minimum}$$

$$\frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} < 0, \text{ a maximum}$$

$$\frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} = 0, \text{ an inflection point}$$

In Figure 4 a function Y = f(X) is shown. When the value of X is X_A, the function has a maximum. The slope of the function ($\frac{dY}{dX}$) at point A is 0. The second derivative ($\frac{d^2Y}{dX^2}$) is negative. In Figure 4, when the value of X is X_B, the function is at a minimum at point B. The derivative is 0 and the second derivative is positive or greater than O. Between points A and B, the function changes from "decreasing at an increasing rate" to "decreasing at a decreasing rate." This point of transition is an inflection point. Technically there is no tangent that can be constructed at the inflection point.

Figure 4



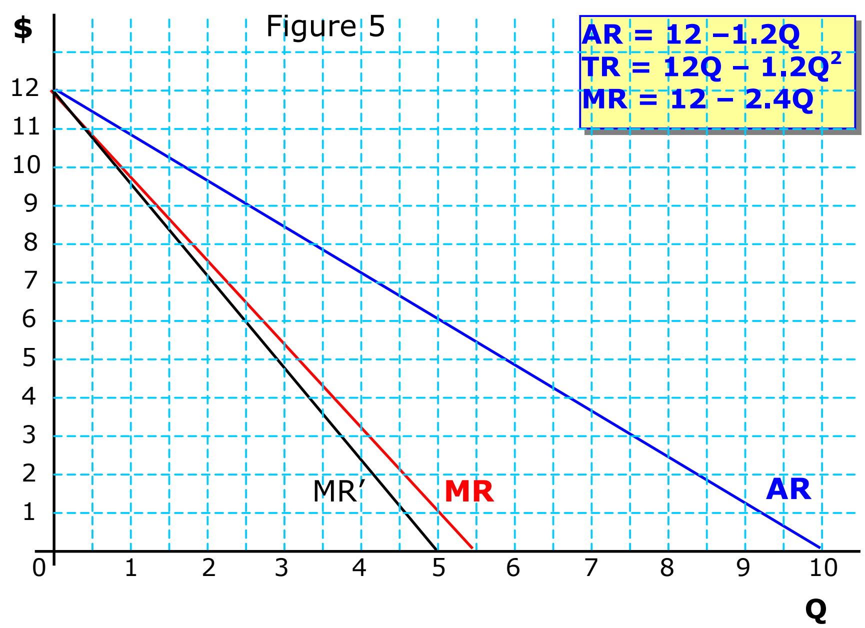
Integrals

An integral can be conceptualized as the "area under a curve," or the inverse of a derivative. Integration is the process of reconstructing a function from is derivative. In Figure 5, average revenue (AR) and 2 marginal revenue functions (MR) are shown. The MR function was developed by subtraction, i.e. $\frac{\Delta TR}{\Delta Q}$. The MR' function was constructed with $\frac{dTR}{dQ}$. Note that the MR function calculated by subtraction is a "half click off".

Table 5

Q	AR=P	TR	MR
0	12	0	-
1	10.8	10.8	10.8
2	9.6	19.2	8.4
3	8.4	25.2	6
4	7.2	28.8	3.6
5	6	30.0	1.2
6	4.8	28.8	-1.2
7	3.6	25.2	-3.6
8	2.4	19.2	-6
9	1.2	10.8	-8.4
10	0	0	-10.8

Figure 5



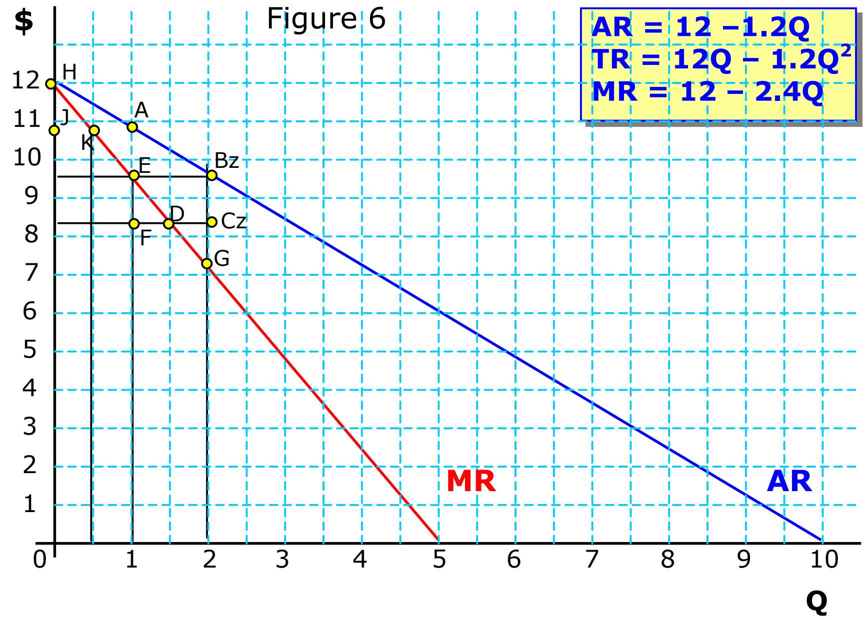
The MR function calculated by subtraction is different from the MR' calculated by derivatives. This occurs because when subtraction is used, the changes are in discrete units and with derivatives they are continuous. When Q changes from 0 to 1, TR changes from 0 to 10.8. When using a table to present the information, the convention is to assign the 10.8 value for the ΔTR to the row labeled 1. Technically, it should be between row 0 and 1. The value of MR (calculated by subtraction) will be graphed half way between the integers. When calculating the MR′ by derivative, the slope of the function can be evaluated at the value of Q (the independent variable) in each row. To illustrate the idea of an integral, discrete changes in the independent variable are used. As the value of Q changes from 0 to1 (ΔQ = 1), the ΔTR is 10.8. The "marginal revenue" of the first unit can be shown as a rectangle on Figure 6.

In Figure 6 when Q=1,

- P = 10.8
- TR=PQ=10.8

TR is 0 when Q=0, TR_1 is 10.8 when Q=1, therefore MR_1= 10.8. Constructing rectangle 01AJ =TR=PQ. TR and MR are the same for the first unit.

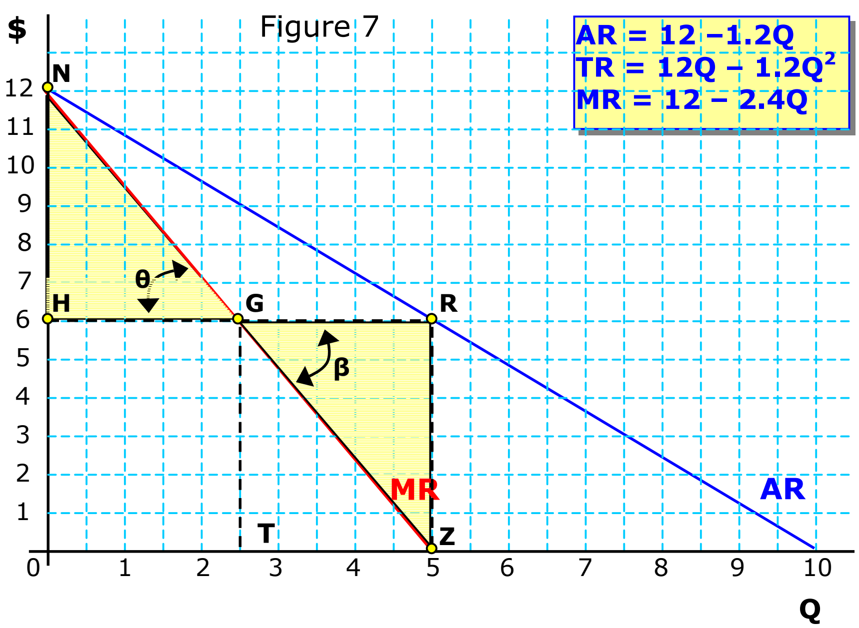
Alternatively, TR is the integral of the MR or area 01EH. Area 01EH is the same as area 01AJ because triangle HJK is congruent with triangle EAK (angle-side-angle equals angle-side-angle).



This can also be stated,

- TR = PQ = (10.8)1
- $TR=\int_0^1 dQ=TR(1)$

If Q is 2, TR = 19.2 making MR_2 (marginal revenue of the second unit) 8.4. TR can be calculated by PQ=(9.6)2=19.2. Alternatively, using MR, TR is the area 02GH in Figure 6. This is because triangle EFD is the same as triangle GCE. By the addition of the marginals (10.8 + 8.4 = 19.2) the TR can be obtained. As the ΔQ→0, the triangles become smaller and smaller and the sum of the marginals becomes the equivalent of an integral.



In Figure 7 the relationship of TR, AR and MR is illustrated again. When AR is linear, MR will fall at 2X the rate as AR, i.e. the slope of MR is twice the slope of AR. This means that the MR will bisect the angle formed at the Y-intercept of the AR (point N). The line segment HG will equal the line segment GR. The MR will have an X-intercept at ½ the AR intercept on the Q or X-axis.

Triangle HGN is the same as Triangle RGZ. Angle **θ** is equal to angle **β**. Line segment HG = GR and the angles at R and H are right angles. In geometry the two triangles are congruent because "angle-side-angle = angle-side-angle." This makes the area under the MR (0

Integrals can be used to find the area under a curve. Given MC and MR the areas under each curve can be obtained to give TC and TR. The difference between the two areas (TR and TC) would be profit. An integral may be used to estimate total benefits (TB) given marginal benefits. The area under a demand function above the market price is consumer surplus and is used as an estimate of consumer welfare.

Summary

The "Marginalist Revolution" of the late 19th century was the result of the application of calculus to economic phenomena. The word "marginal" refers to the rate of change in a dependent variable "caused" by a change in an independent variable. Marginal can refer to the rate of change as the slope of a line $\left(\frac{\Delta Y}{\Delta X}\right)$ or the slope of a tangent to a nonlinear function at a point $\left(\frac{dY}{dX}\right)$. The concept of a "marginal" is fundamental to choosing among alternatives.

An integral is useful to measure areas and volumes. Given a marginal value a total can be calculated. Integrals measure the area under a curve. Consumer and producer surplus, total revenue, total cost, total profits, etc can be estimated using integrals.

Example Problems

1) Give a demand function Q = 600 – 5P, find the price that will result in the maximum total revenue (TR) and what is the maximum TR possible?

Since TR is defined as the product of price and quantity and marginal revenue is $\frac{dTR}{dQ}$ the expression will be converted from Q as a function of P to P as a function of Q.

$Q = 600 - 5P$

subtract 600 from both sides

$-600 + Q = -5P$

divide both sides by -5

$\frac{-600}{-5} + \frac{1}{-5}Q = \frac{-5P}{-5}$

so

$120 - .2Q = P$

or

$P = 120 - .2Q$

Now that we have P = 120 -.2Q, TR can be calculated

$P = 120 - .2Q$

$TR = PQ$

By substitution, $TR = (120 - .2Q)Q = 120Q - .2Q^2$

To find the Q where TR is a maximum, take the first derivative and set it equal to 0.

$TR = 120Q - .2Q^2$

the first derivative of TR with respect to Q is

$$\frac{dTR}{dQ} = 120 - .4Q$$

Setting $\frac{dTR}{dQ} = 0$

$120 - .4Q = 0$

Add .4Q to both sides

$120 = .4Q$

$Q = 300$

Since Q = 300, the value of P that will maximize TR can be found by substituting 300 for Q in the demand equation P = 120 -.2Q.

$P = 120 - .2Q = 120 - .2(300) = 60$

Where **P = 60**, TR will attain a maximum value of $18,000.

2) Given a Total Cost (TC function) $TC = 400Q – 5Q^2 + .03Q^3$, find the output that will minimize the cost per unit. What is the minimum cost per unit?

The cost per unit is the same as the Average Cost (AC), which is $\frac{TC}{Q}$.

Calculate AC,

$TC = 400Q - 5Q^2 + .03Q^3$

Divide TC by Q

$$\frac{TC}{Q} = AC = \frac{400Q}{Q} - \frac{5Q^2}{Q} + \frac{.03Q^3}{Q} = 400 - 5Q + .03Q^2$$

To find the Q where AC is a minimum, Take the first derivative of AC and set it equal to 0.

$$\frac{dAC}{dQ} = -5 + .06Q$$

set $\frac{dAC}{dQ} = 0$

$-5 + .06Q = 0$

Add 5 to both sides

$.06Q = 5$

divide both sides by .06

$$Q = \frac{5}{.06} = 83.333$$

Since Q = 83.33, the minimum average cost (AC_{min}) can be calculated by substitution.

$AC = 400 - 5Q + .03Q^2 = 400 - 5(83.33) + .03(83.33)^2$

AC = $191.67