Measuring the Condensate Fraction of Rapidly Rotating Trapped Boson Systems: Off-Diagonal Order from the Density Profile

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We demonstrate a direct connection between the density profile of a system of ultracold trapped bosonic particles in the rapid-rotation limit and its condensate fraction. This connection can be used to probe the crossover from condensed vortex-lattice states to uncondensed quantum-fluid states that occurs in rapidly rotating boson systems as the particle density decreases or the rotation frequency increases. We illustrate our proposal with a series of examples, including ones based on models of realistic finite trap systems, and comment on its application to freely expanding boson density profile measurements.

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When a system of coherent trapped bosons is brought to equilibrium in a rotating frame of reference, vortices are induced in the order parameter of its condensate and centrifugal forces lead to radial expansion. The influence of the rotating frame may be described mathematically by introducing an effective magnetic field oriented along the axis of rotation and reducing the radial confinement strength. Taking this point of view, very rapidly rotating boson systems enter the quasi-two-dimensional (2D) strong-field quantum-Hall regime in which interactions are always important. Many-fermion systems in the quantum-Hall regime have been thoroughly studied over the past 20 years and have exhibited a wide variety of subtle and novel behaviors, many of which were completely unanticipated by theory [1]. This history provides a powerful motivation for creating boson quantum-Hall systems. Most current experiments in rotating Bose-Einstein systems create condensates in which the effective magnetic field is relatively weak and axial-direction quantum confinement effects are negligible when compared to interaction strengths. For such condensates, the nucleation of vortices and the formation of vortex lattices, together with their decay and evolution, have been observed experimentally and studied theoretically by several groups [2–8].

The limit of quasi-2D bosons in the rapid-rotation (or equivalently, quantum-Hall) regime has so far been studied only theoretically [9–14]. The physics of trapped bosons in the quantum-Hall regime is enriched by the near degeneracy of single-particle energy levels, which enhances the importance of condensate quantum fluctuations [9], reduces the condensate fraction, and eventually, at low boson densities, leads to quantum-fluid states [12,13] that are expected to have unusual properties analogous to those that have been uncovered in studies of the fermion quantum-Hall effect. One of the major stumbling blocks in contemplating studies of bosons in the quantum-Hall regime has been the lack of a reliable tool to directly measure the degree of Bose-Einstein condensation and the accompanying long-range many-boson phase coherence. Unlike its familiar zero-rotation cousin, a vortex-lattice condensate in the quantum-Hall regime does not have a characteristic velocity distribution that allows it to be identified from the *velocity* distribution of the released system. In this paper, we show that in the quasi-2D rapid-rotation limit there is, however, a surprisingly simple relationship between diagonal elements of the many-boson density matrix and the off-diagonal order responsible for coherence. The condensate fraction can be probed simply by measuring the spatial distribution of the boson *density*.

Before explaining the simple but powerful relation that we propose be exploited, we first discuss its regime of validity. A system of N interacting bosons of mass M in a cylindrical trap (with radial and axial trap frequencies Ω_r and Ω_z) that is rotating with angular velocity $\Omega\hat{\mathbf{z}}$ is well described by the rotating-frame Hamiltonian [15]

$$\mathcal{H} = \sum_{i=1}^{N}\bigg\{\frac{(\mathbf{p}_i - M\Omega\hat{\mathbf{z}}\times\mathbf{r}_i)^2}{2M} + \frac{M}{2}[(\Omega_r^2-\Omega^2)(x_i^2+y_i^2)+\Omega_z^2 z_i^2]\bigg\} + \sum_{i<j=1}^{N} V(\mathbf{r}_i-\mathbf{r}_j), \quad (1)$$

where $V(\mathbf{r}) = g\delta(\mathbf{r})$ is a hard-core interaction potential of strength $g = 4\pi\hbar^2 a_s/M$, and a_s is the s-wave scattering length. This Hamiltonian is equivalent to that of a system of charge-Q bosons with weakened radial confinement, under the influence of a magnetic field $\mathbf{B} = (2M\Omega/Q)\hat{\mathbf{z}}$, with cyclotron frequency $\Omega_c = QB/M = 2\Omega$ and magnetic length $\ell = \sqrt{\hbar/(2M\Omega)}$. The Hilbert space of this system is spanned by the axial (z direction) quantum states and the various Landau-level states produced by the effective magnetic field. For Ω close to Ω_r, the effective confinement potential becomes very weak, which causes the bosons to spread out radially and, in the limit of interest to us here (see below), allows all bosons to be accommodated in the lowest-energy axial quantum state.

In the quasi-2D limit, the boson system can be described by a 2D model whose short-range interactions have strength $g_{2D} = \hbar^2 a_s \sqrt{8\pi}/M l_z$, where $l_z = \sqrt{\hbar/M\Omega_z}$. The areal density of bosons in such a trap can be estimated using a 2D Thomas-Fermi (TF) approximation in which $M(\Omega_r^2 - \Omega^2)r^2/2 + g_{2D}n_{2D}$ is constant inside the system [16]. Since the bosons can form rather uniform-density clouds (of the type favored by short-range interactions) entirely within the lowest Landau level (LLL), there is little incentive for them to occupy higher kinetic-energy Landau levels, unless the typical interaction strength $g_{2D}n_{2D}$ is much larger than the Landau-level splitting $2\hbar\Omega$. The TF approximation boson Landau-level filling factor is given by $\nu(r) \equiv 2\pi\ell^2 n_{2D}(r) = \nu_0(1 - r^2/R^2)$, where $R^2 = \sqrt{4g_{2D}N/\pi M(\Omega_r^2 - \Omega^2)}$, and $\nu_0^2 = \sqrt{2\pi} l_z N(\Omega_r^2/\Omega^2 - 1)/4a_s$. The parameter

$$\gamma_{\mathrm{LL}} \equiv \frac{g_{2D}n_{2D}(r=0)}{2\hbar\Omega} = \sqrt{\frac{(\Omega_r^2/\Omega^2 - 1)a_s N}{l_z\sqrt{2\pi}}} \tag{2}$$

characterizes the degree of Landau-level mixing induced by interactions. Neglecting boson Landau-level mixing, as we do in the rest of the paper, is a good approximation provided that γ_{LL} is not large. Similarly, neglecting mixing of higher axial energy eigenstates is a good approximation provided that $\gamma_z = 2\gamma_{\mathrm{LL}}\Omega/\Omega_z$ is not too large. In mean-field theory, the effect of a finite value of γ_z is to weaken axial confinement at the center of the trap and therefore to effectively weaken g_{2D}. The conclusions we reach below would not be altered by a position-dependent effective interaction strength. We therefore expect our results to apply for values of γ_{LL} and γ_z that are substantially larger than 1, although more detailed calculations will be required to establish the range of the rapid-rotation limit more quantitatively. Values of γ_{LL} and γ_z that are not much larger than unity are already within reach of current experimental designs [17].

Condensation of a system of trapped bosons (we consider here only the zero-temperature case) is discussed most generally in terms of the spectrum of its one-particle density matrix (ODM), whose position representation is related to the many-boson wave function by

$$\rho_1(\mathbf{r},\mathbf{r}') \equiv N\int d\mathbf{r}_2,\ldots,d\mathbf{r}_N \Psi_N^*(\mathbf{r},\mathbf{r}_2,\ldots,\mathbf{r}_N)\Psi_N(\mathbf{r}',\mathbf{r}_2,\ldots,\mathbf{r}_N) = \langle\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r}')\rangle, \tag{3}$$

where $\rho_1(\mathbf{r},\mathbf{r}) = n(\mathbf{r})$ is the boson density, and the trace of the ODM is the total particle number, N. If one or more of the eigenvalues, λ_i, of the ODM is extensive (i.e., $\lambda_i \propto N$), then the system is Bose condensed [15]. In a particle-conserving system, the ratio

$$\beta = \frac{\mathrm{Tr}(\rho_1^2)}{(\mathrm{Tr}\rho_1)^2} \tag{4}$$

will be of order unity *only* if the system is Bose condensed and the condensate fraction will be $\sqrt{\beta}$, up to a correction of order $1/N$.

The many-body physics of quasi-2D boson systems in a magnetic field simplifies in the rapid-rotation limit because the many-boson Hilbert space can be projected to the LLL. These simplifications are often most conveniently captured in the symmetric gauge, where they imply that many-particle wave functions must have the form $\Psi_N(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N) = f[z]\exp(-\sum_k |z_k|^2/4\ell^2)$, where $f[z]$ is analytic in each of the N complex bosonic coordinates $z_k = x_k + iy_k$. The analyticity property has played a key role in achieving an understanding of many-fermion physics in the quantum-Hall regime, for example, in Laughlin's recognition [18] that incompressible states would occur at certain Landau-level filling factors. Because of this property, the ODM is completely specified by the functional form of its diagonal matrix element. For fermions, this property leads [19] to Hartree-Fock exchange energies that depend only on the particle density; for bosons, it has more fundamental implications, as we now discuss.

It follows from $\Psi_N(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N)$ that

$$\rho_1(\mathbf{r}',\mathbf{r}) = n[z^*,z']\exp\left(-\frac{|z|^2}{4\ell^2} - \frac{|z'|^2}{4\ell^2} + \frac{zz'^*}{2\ell^2}\right), \tag{5}$$

where $n[z^*,z]$ is the boson density expressed as a function of $z = x + iy$ and $z^* = x - iy$. Equation (5) can be used to express β in terms of the 2D Fourier transform of the boson density: $n[\mathbf{r}] = n[z^*,z] = \int d\mathbf{q}\, n(\mathbf{q})\exp[i(q^*z + qz^*)/2]/(4\pi^2)$, where $q = q_x + iq_y$. We find that

$$\beta = \frac{\ell^2}{2\pi N^2}\int d\mathbf{q}|n(\mathbf{q})|^2 e^{q^2\ell^2/2}. \tag{6}$$

We propose that Eq. (6) be used to measure the condensate fraction of rapidly rotating quasi-2D Bose-Einstein condensates (BEC)'s.

To understand the strengths and weaknesses of this condensate measure as an experimental probe, it is helpful to discuss several different examples. We first consider the case of a rapidly rotating fully condensed BEC that contains a single vortex; for typical traps and interaction strengths, such a state would occur only for boson particle numbers that are too small to study at the present time. In this case, the Gross-Pitaevskii (GP) order parameter [15] is $\phi(z) = \sqrt{N/(4\pi\ell^4)}z\exp(-|z|^2/4\ell^2)$ which leads to $n(q) = N(1 - q^2\ell^2/2)\exp(-q^2\ell^2/2)$, and yields $\beta = 1$, as expected.

A qualitatively similar but more realistic case is illustrated in Fig. 1. The upper panel of this figure shows the mean-field-theory 2D boson density profile, obtained by solving the GP equations following Refs. [10,20], while

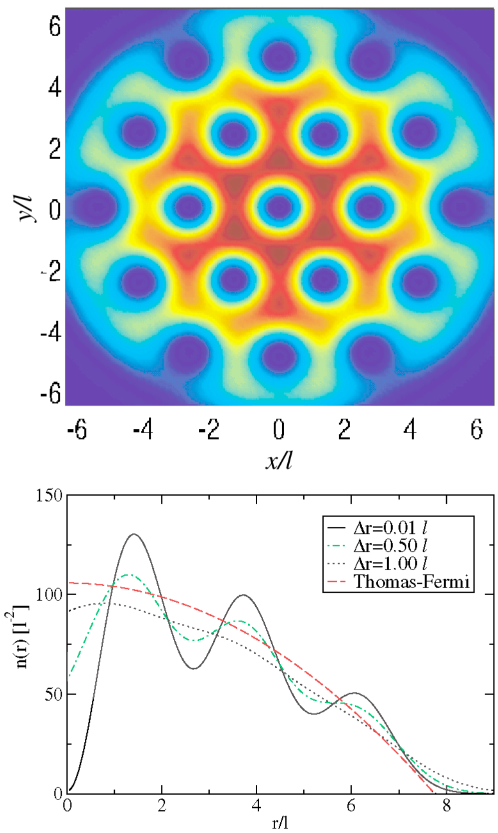

FIG. 1 (color online). Upper panel: Density profile for the ground state of a rapidly rotating boson system with $\Omega_r/\Omega = 1.04$, $g_{2D}M/\hbar^2 = 0.0058$, and $N = 10\,000$. These parameters are similar to ones that can be achieved by current experimental systems. For these parameters, the optimal number of vortices is 19. Lower panel: Angle-averaged boson filling-factor profile with radial broadening Δr. The long-dashed line in this figure is the TF approximation boson filling-factor profile for the same parameters.

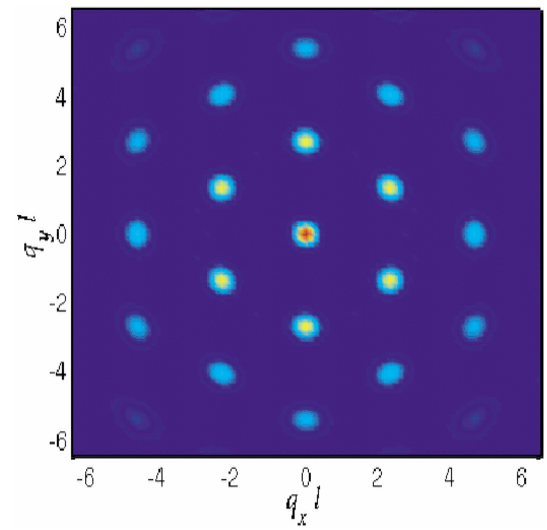

FIG. 2 (color online). $|n(q)|^2 \exp(q^2\ell^2/2)$ for the density profile shown in Fig. 1. The axes are those of the 2D wave vector, multiplied by the effective magnetic length, ℓ.

allowing the trap's rotational symmetry to be broken and optimizing with respect to the number of vortices, for the parameters $\Omega_r/\Omega = 1.04$, $g_{2D}/(\hbar^2 M) = 0.0058$, and $N = 10\,000$. These parameters were chosen to closely match those of current experimental systems [21]. The lower graph in Fig. 1 shows the profile of the angle-averaged boson filling factor. Under these circumstances, we expect [9] that the condensate fraction will be large, at least in the limit of zero temperature. When smoothed radially over a length $\sim\ell$, the boson density profile agrees reasonably well with that predicted by TF theory, illustrating the dominant role of interactions in determining the course-grained density profile, even in the rapid-rotation limit. In Fig. 2, we show the integrand of Eq. (6) that is involved in calculating β, for the configuration shown in Fig. 1. In practice however, only the first few rings of Fig. 2 can be resolved in current experiments and a finite wave-vector cutoff must be introduced [17]. In this case, two different estimates of β might prove useful. A lower bound β_- is obviously obtained by calculating β using an experimentally constrained momentum cutoff $q_{\max}$. A separate estimate of β, which is more likely to be useful in practice for current experimental systems, is motivated by observing that in the case of a quasi-2D (untrapped) condensate each Brillouin zone in momentum space contributes equally to $\int d^2\mathbf{q}|n(\mathbf{q})|^2 \exp(|q^2|\ell^2/2)$. It follows that the ratio of diffraction peak strengths integrated over Brillouin zones centered on $\mathbf{q} \neq 0$ (condensate contributions only) and $\mathbf{q} = 0$ (condensate plus normal fluid contributions) can be used to estimate β. The efficacy of this approach could be checked by comparing estimates from different shells [22]. For the example of Fig. 2, we obtain $\beta_- \approx 0.27, 0.57, 0.83$ for cutoffs of $q_{\max}\ell = 4, 6, 9$, and, using the first shell for the second estimate we obtain $\beta \approx 0.93$; i.e., all show clear evidence of Bose-Einstein condensation, albeit with an inaccurate value of the condensate fraction. (The condensate fraction is, of course, always one for GP approximation density profiles.) As this calculation illustrates, we do not anticipate that our method will be especially useful for measuring small deviations of the condensate fraction from unity in strongly condensed systems. We do, however, expect that meaningful measurements of β will be possible when the condensate fraction is strongly reduced due to either thermal or quantum fluctuations, and anticipate that strong reductions will occur at available temperatures even at very high values of ν_0.

We next examine the quantum-fluid regime in which rotational symmetry is not broken and the 2D density profile should be given approximately by the TF approximation expression, $n(r) = \nu_0(1 - r^2/R^2)/(2\pi\ell^2)$. For $R/\ell \gg 1$, we find that $\beta \approx (2/3)\nu_0/N$, after introducing an ultraviolet cutoff to remove an artificial cusp in the TF density. This result illustrates the ability of our condensate measure to capture the property that, in the rapid-rotation limit, BEC can occur only when accompanied by broken rotational symmetry. Incompressible boson states [12–14] at very small filling factors are expected to produce more uniform-density profiles, but they also have $\beta \sim \nu/N$ and are thus uncondensed.

As a final example, we consider a state with two independent condensates, A and B, with a density profile $n(\mathbf{r}) = \lambda_A n_A(\mathbf{r}) + \lambda_B n_B(\mathbf{r})$, where $\lambda_A + \lambda_B = 1$. For the case in which species A is in an uncondensed constant-density state and species B is a large vortex-lattice condensed state, we obtain $\beta = \lambda_B^2/N + (2\lambda_A + \lambda_A^2)\nu/N$. Therefore, β will be of order unity even if only one of the species is condensed.

Another experimental issue that might sometimes arise in the measurement of β is the determination of the boson density profile from free-expansion images. As shown in Refs. [10,23], the evolution of the order parameter under 2D free expansion (i.e., when the confinement in the z direction is maintained) is given by $\phi_{\mathrm{lab}}(\mathbf{r}, t) = \exp[-i(\Omega L_z - \mu)\tau(t)/\hbar]\phi_{\mathrm{rot}}[\mathbf{r}/\lambda(t)]/\lambda(t)$, for a system obeying the time-dependent GP equation. Here ϕ_{lab} is the order parameter viewed in the laboratory frame, and $\phi_{\mathrm{rot}}(\mathbf{r})$ is the order parameter before the trap has been turned off (obtained from the time-independent GP equation in the initial rotating frame), $\lambda(t) = \sqrt{1 + \Omega_r^2 t^2}$, and $\Omega_r \tau(t) = \arctan(\Omega_r t)$. The only assumption used to obtain this result, aside from ignoring interactions with the thermal cloud (which is justified by the short expansion times involved), is the validity of the time-dependent GP equation. In the quantum-Hall regime bosons can enter a regime in which quantum fluctuations render these assumptions invalid, particularly when regions of the condensate near the edge that have low local filling fractions become important. It is possible that such low-density regions could evolve under free expansion differently than the higher filling fraction regions near the center. Hence, whenever β is no longer of order unity, the analysis of the boson density profile using free-expansion measurements is suspect and one must resort to nondestructive (i.e., unexpanded) measurements of the density profile. Given the large radial expansion expected in this regime of interest here, however, it may be possible to resolve a sufficiently large number of shells of reciprocal space $n(\mathbf{q})$ diffraction peaks to apply the method we propose.

In summary, we propose a new experimental tool to explore the transition between a vortex-lattice Bose-Einstein condensate and an uncondensed quantum-fluid state in rapidly rotating boson systems. This new tool derives from a deep connection between the density profile of a system of ultracold bosonic particles in the rapid-rotation limit and their degree of condensation. It applies in the quasi-2D rapid-rotation limit that current experiments are approaching.

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