1 Objectives

The objectives of this laboratory experiment are:

- To investigate the time response of a first-order circuit to a step input and to determine its characteristics.
- To investigate the time response of an underdamped second-order circuit to a step input and to determine its characteristics.

2 Theory

Consider the transfer function of a first-order circuit represented by the block diagram shown in Figure 1. This transfer function is said to be normalized if $K = \omega_o$. Otherwise, it is said to be unnormalized.

For a unit step change in the input voltage,

$$v_i(t) = u(t) \implies V_i(s) = \frac{1}{s}$$  \hspace{1cm} (1)

the circuit output response is

$$V_o(s) = H(s)V_i(s) = \frac{K}{s + \omega_o} \times \frac{1}{s} = \frac{K}{s(s + \omega_o)}$$

Expanding this Laplace fraction into a partial expansion of elementary terms, we obtain

$$V_o(s) = \frac{K/\omega_o}{s} - \frac{K/\omega_o}{s + \omega_o}$$ \hspace{1cm} (2)

whose inverse Laplace transform is

$$v_o(t) = \frac{K}{\omega_o} (1 - e^{-\omega_o t}) = \frac{K}{\omega_o} (1 - e^{-t/\tau})$$ \hspace{1cm} (3)

where

$$\tau = \frac{1}{\omega_o}$$ \hspace{1cm} (4)
is the time constant of the circuit. This time constant can be interpreted as the time it takes for the output voltage $v_o(t)$ to increase from zero to

$$v_o(\tau) = \frac{K}{\omega_o}(1 - e^{-1}) = 0.632 \frac{K}{\omega_o}$$

or about 63.2% of its final or steady-state value

$$V_{o,ss} = \frac{K}{\omega_o}$$

The steady error of the output with the unit step input is

$$E_{ss} = V_{o,ss} - V_i = \frac{K}{\omega_o} - 1$$

![Second-Order Transfer Function](image)

Figure 2: Second-Order Transfer Function

Consider the transfer function of a second-order circuit represented by the block diagram shown in Figure 2. This transfer function is said to be normalized if $K = \omega_n^2$. Otherwise, it is said to be unnormalized. In the following discussion, assume a normalized transfer function with $K = \omega_n^2$ and expressed in a standard notation as

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2}$$

where $\omega_n$ is called the natural frequency and has units of rad/s and $\zeta$ is called the damping factor and is unitless. The physical interpretation of $\omega_n$ and $\zeta$ will be explained shortly.

For given values of $\omega_n$ and $\zeta$ and a step change in the input voltage $v_i(t)$, the transient performance of the output voltage will depend on the poles of the transfer function $H(s)$, that is, on the roots of the denominator of $H(s)$. These roots will be either real or complex depending on the sign of the discriminant

$$\Delta = (2\zeta \omega_n)^2 - 4\omega_n^2 = 4\omega_n^2(\zeta^2 - 1)$$

The second-order circuit is said to be overdamped if the discriminant $\Delta$ is positive, that is, if $\zeta > 1$. In this case, the transfer function $H(s)$ has two real negative poles given by

$$s_{1,2} = -\sigma_{1,2} = -\zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$$

The second-order RLC circuit is said to be critically-damped if the discriminant $\Delta$ is zero, that is, if $\zeta = 1$. In this case, the transfer function $H(s)$ has a double real negative pole given by

$$s_{1,2} = -\sigma, -\sigma = -\omega_n, -\omega_n$$

The second-order circuit is said to be underdamped if the discriminant $\Delta$ is negative, that is, if $0 \leq \zeta < 1$. In this case, the transfer function $H(s)$ has two complex conjugate poles with negative real parts given by

$$s_{1,2} = -\sigma \pm j\omega = -\zeta \omega_n \pm j\omega_n \sqrt{1 - \zeta^2}$$
The condition of the circuit (overdamped, critically-damped, or underdamped) is specified in Figure 3 by the location of the poles of the transfer function $H(s)$ in the complex plane. Notice that in the case of the underdamped circuit, the magnitude of each of the complex conjugate poles is equal to the natural frequency $\omega_n$ since

$$|s_{1,2}| = \sqrt{\sigma^2 + \omega^2} = \sqrt{(\zeta \omega_n)^2 + \omega_n^2(1 - \zeta^2)} = \omega_n$$

(12)

As to the damping factor $\zeta$, notice that

$$\zeta = \frac{\zeta \omega_n}{\omega_n} = \frac{\zeta \omega_n}{\sqrt{(\zeta \omega_n)^2 + \omega_n^2(1 - \zeta^2)}} = \sin \theta$$

(13)

where $\theta$ is the angle of a complex pole with respect to the imaginary axis. A condition of zero damping ($\zeta = 0$) would result in two purely imaginary conjugate poles on the $j\omega$-axis since $\theta = 0$.

The circuit response to a unit step change in the input voltage $V_i(s)$ is given by

$$V_o(s) = H(s)V_i(s) = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} \times \frac{1}{s} = \frac{\omega_n^2}{s(s^2 + 2\zeta \omega_n s + \omega_n^2)}$$

Consider first a critically-damped circuit for $\zeta = 1$. For a unit step input in the input voltage,

$$v_i(t) = u(t) \implies V_i(s) = \frac{1}{s}$$

(14)

the output voltage $V_o(s)$ is given by

$$V_o(s) = \frac{\omega_n^2}{s(s + \omega_n)^2} = \frac{1}{s} - \frac{1}{s + \omega_n} - \frac{\omega_n}{(s + \omega_n)^2}$$

(15)

Taking the inverse Laplace transform of $V_o(s)$, we find that the complete transient response of the output voltage is given by

$$v_o(t) = (1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t})$$

for $t \geq 0$.

(16)

Figure 4(a) shows the response of $v_o(t)$. The final (steady-state) value of $v_o(t)$ is $V_{o,ss} = 1$ and does not have a steady-state error. Notice that $v_o(t)$ remains always under unity, that is, it does not have an overshoot whatsoever.

An overdamped response which occurs for $\zeta > 1$ is similar in shape to the critically-damped case and will not possess an overshoot with two different real negative poles. Therefore, it will not be
discussed further here.

Next, consider an underdamped circuit for $0 \leq \zeta < 1$. For a unit step in the input voltage,

$$v_i(t) = u(t) \implies V_i(s) = \frac{1}{s}$$

the output voltage $V_o(s)$ is given by

$$V_o(s) = \frac{\omega_n^2 V_i(s)}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{1}{s} - \frac{(s + 2\zeta\omega_n)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Taking the inverse Laplace transform of $V_o(s)$, we find that the complete transient response of the output voltage is given by

$$v_o(t) = (1 - e^{-\zeta\omega_n t} \cos(\omega_n \sqrt{1 - \zeta^2} t) - \frac{\zeta e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}} \sin(\omega_n \sqrt{1 - \zeta^2} t), \quad \text{for} \quad t \geq 0$$

Using the previously-defined angle $\theta$ in Figure 4 and

$$\sin \theta = \zeta$$
$$\cos \theta = \sqrt{1 - \zeta^2}$$

we can rewrite Equation (20) as

$$v_o(t) = \left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}} \cos(\omega_n \sqrt{1 - \zeta^2} t - \theta) \right], \quad \text{for} \quad t \geq 0$$

Figure 4: (a) Critically-Damped and (b) Underdamped Circuit Responses

This last expression can be conveniently rewritten as

$$V_o(s) = \frac{1}{s} - \frac{(s + \zeta\omega_n)}{s(s + \zeta\omega_n)^2 + \omega_n^2 (1 - \zeta^2)} - \frac{\zeta}{\sqrt{1 - \zeta^2}} \times \frac{V_{dc} \omega_n \sqrt{1 - \zeta^2}}{(s + \zeta\omega_n)^2 + \omega_n^2 (1 - \zeta^2)}$$

Taking the inverse Laplace transform of $V_o(s)$, we find that the complete transient response of the output voltage is given by

$$v_o(t) = (1 - e^{-\zeta\omega_n t} \cos(\omega_n \sqrt{1 - \zeta^2} t) - \frac{\zeta e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}} \sin(\omega_n \sqrt{1 - \zeta^2} t), \quad \text{for} \quad t \geq 0$$

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$$\sin \theta = \zeta$$
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$$v_o(t) = \left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}} \cos(\omega_n \sqrt{1 - \zeta^2} t - \theta) \right], \quad \text{for} \quad t \geq 0$$
Figure 4(b) shows the response of $v_0(t)$ for a unit step input $v_i(t) = u(t)$. The final (steady-state) value of $v_0(t)$ is $V_{o,ss} = 1$. Notice that $v_0(t)$ is oscillatory in shape because of the complex nature of the poles of $H(s)$. The transient frequency $\omega$ is defined as

$$\omega = \omega_n \sqrt{1 - \zeta^2}$$

(24)

This frequency can be found in Figure 4(b) by measuring the transient period $T$ between two successive peaks. Other performance measures of the underdamped response are as follows:

The maxima and minima (peaks) of the underdamped circuit response occur when

$$\frac{dv_0}{dt} = \frac{\omega_n}{\sqrt{1 - \zeta^2}} \sin(\omega_n \sqrt{1 - \zeta^2} t) = 0 \implies \omega_n \sqrt{1 - \zeta^2} = 0, \pi, 2\pi, \ldots$$

(25)

The peak time $T_p$ is defined as the time of the first maximum peak and occurs when

$$\omega_n \sqrt{1 - \zeta^2} T_p = \pi \implies T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}$$

(26)

The peak value at time $T_p$ is equal to

$$M_p = (1 + e^{-\zeta \omega_n T_p}) = (1 + e^{-\pi \zeta \sqrt{1 - \zeta^2}})$$

(27)

Thus the maximum percent overshoot PO is equal to

$$PO(\%) = (M_p - 1) \times 100\% = 100e^{-\pi \zeta \sqrt{1 - \zeta^2}}$$

(28)

and it depends only on the damping factor $\zeta$ in the circuit.

### 3 Equipment

- Agilent DSO5014A Digital Storage Oscilloscope
- Agilent 33220A Function/Arbitrary Waveform Generator
- Fluke 115 True RMS Multimeter

### 4 Procedure

**Part A: Transient Response of a First-Order Circuit**

1. Build the op-amp circuit of Figure 5. Measure the capacitor value $C_4$ using a shared RLC meter and measure the resistor value $R_4$ using the benchtop multimeter.

<table>
<thead>
<tr>
<th></th>
<th>$f$ (kHz)</th>
<th>$C_4$ (nF)</th>
<th>$R_4$ (kΩ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nominal</td>
<td>1</td>
<td>10</td>
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</tr>
<tr>
<td>Measured</td>
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2. Apply a 20-Hz, 1-V peak-to-peak square wave with 0.5 V DC offset to the op-amp circuit. Using the cursors on the oscilloscope, measure the time constant $\tau$ and the steady-state value $V_{o,ss}$ of the first-order circuit.
Part B: Transient Response of an Underdamped Second-Order Circuit

1. Build the op-amp circuit of Figure 6. Measure the capacitor value $C_2$ using the shared RLC meter and use precision resistors.

<table>
<thead>
<tr>
<th>$f$ (kHz)</th>
<th>$C_2$ (nF)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nominal</td>
<td>1</td>
</tr>
<tr>
<td>Measured</td>
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</tbody>
</table>

2. Apply a 20-Hz, 1-V peak-to-peak square wave with 0.5 V DC offset to the op-amp circuit. Observe the underdamped circuit response. Notice that the square wave allows us to charge and discharge the capacitor voltage so that we can continuously display the transient response on the oscilloscope.

3. Measure the peak time $T_p$, peak value $M_p$, and rise time $T_r$.

<table>
<thead>
<tr>
<th></th>
<th>$T_p$ (ms)</th>
<th>$M_p$ (V)</th>
<th>$T_r$ (ms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nominal</td>
<td>3.63</td>
<td>1.16</td>
<td>-</td>
</tr>
<tr>
<td>Measured</td>
<td></td>
<td></td>
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</table>
5 Data Analysis and Interpretation

1. In Part A, compare the time constant \( \tau \) measured with the oscilloscope with the expected (theoretical) value of \( \tau = \frac{R_4 C_4}{2} \) obtained by substituting the measured values of \( R_4 \) and \( C_4 \).

<table>
<thead>
<tr>
<th>Measured</th>
<th>Predicted</th>
<th>Percent Error (%)</th>
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</thead>
<tbody>
<tr>
<td>( \tau ) (ms)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. In Part B, compute \( \zeta \) using Equation (27) and the measured \( M_p \).

3. Substitute \( \zeta \) into Equation (26) and solve for \( \omega_n \) using the measured \( T_p \).

4. Compute the poles of the underdamped transfer function by solving the quadratic equation

\[
D(s) = s^2 + Ks + K^2 = 0 \implies s_{1,2} = \frac{-K \pm jK\sqrt{3}}{2}
\]  

where \( K = \frac{1}{R_4 C_4} \). From these solutions, show that \( \zeta = 0.5 \) and \( \omega_n = K \), and compare these new values of \( \zeta \) and \( \omega_n \) with their previous values computed using the measurements of \( M_p \) and \( T_p \) in Part 2 and Part 3.

<table>
<thead>
<tr>
<th>Measured</th>
<th>Calculated</th>
<th>Percent Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \zeta )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \omega_n ) (rad/s)</td>
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Experiment #9: Transient Response of First- and Second-Order Circuits

Date:

Data Sheet Recorded by:

<table>
<thead>
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<th>Equipment List</th>
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<tbody>
<tr>
<td>Equipment Description</td>
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<tr>
<td>HP/Agilent 54810A Infinium Oscilloscope</td>
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<tr>
<td>HP/Agilent 33120A Function/Arbitrary Waveform Generator</td>
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<td>Fluke 115 True RMS Multimeter</td>
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<td>HP/Agilent E3631A Triple Output Power Supply</td>
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Part A: Transient Response of a First-Order Circuit

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</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$\tau$ (ms)</th>
<th>$V_{o,ss}$ (V)</th>
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Part B: Transient Response of an Underdamped Second-Order Circuit

<table>
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