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Oscillatory pumping wells in phreatic, compressible, and homogeneous aquifers

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Key Points:

- Analytical solution of periodic head field for oscillatory pumping is derived
- The major impact of the water table boundary condition is set forth
- Identification procedure of the parameters is outlined

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Abstract Oscillatory well pumping was proposed recently as a tool for hydraulic tomography. Periodic pumping at a few frequencies is carried out through vertical intervals along the pumping well and the periodic head is measured along a few piezometers. The paper presents an analytical solution for the head field in an unconfined aquifer of finite depth under the common assumptions of a linearized water table condition, different horizontal and vertical constant permeabilities, constant specific storativity and water table drainable porosity, and small well radius to length ratio. The solution provides the expressions of the amplitude and phase of the head as a function of coordinates, frequency, and the problem parameters. The solution simplifies to one pertaining to an upper constant head condition and a rigid aquifer for a wide range of the dimensionless frequency values.

1. Introduction

Pumping tests are one of the most common hydrological procedures used for characterizing aquifer properties. The basis is the solution of the flow equations which renders the head field in space and time; the measured values are used in an inverse mode in order to identify properties (conductivity, storativity).

The standard procedure is to pump at a constant discharge and to measure the drawdown dependence on space and time, for which classical solutions as well as various extensions are available in the literature [see *Kruseman and Ridder*, 1990, for a compendium]. It is emphasized that for usual values of storativity and drainable effective porosity at the water table, the effect of the first is felt only at short times after inception of pumping. Furthermore, due to the linearity of the problem, all the aforementioned solutions are used for recovery as well.

Recently, a method known as 3-D transient hydraulic tomography (3-DTHT) has been proposed for extending the pumping test as an identification tool by *Cardiff and Barrash* [2011] and *Cardiff et al*. [2012, 2013a]. By using packers and accurate sensors, the pumping at constant discharge is carried out from a few subintervals of the well. Similarly the head is measured at various points along a few observation boreholes. While the solution for homogeneous aquifers is an easy adaptation of the previous ones, the more detailed information was employed by *Cardiff and Barrash* [2011] in order to identify the spatially variable conductivity in the covered volume, by a fully numerical solution, and numerical inversion. A recent development of hydraulic tomography known as oscillatory hydraulic tomography (OHT) is by a similar setup, but pumping is of time periodic discharge [see *Cardiff et al*., 2013b; *Bakhos et al*., 2014, and references therein]. One of the advantages of the procedure is that it adds an additional degree of freedom, by varying the frequency.

We set as an objective to attack the problem of oscillatory pumping in unconfined aquifers by regarding the conductivity as random and deriving a semianalytical solution based on a first-order approximation in the log-conductivity variance, in the spirit of *Dagan and Lessof* [2011]. As a first step, we present here the simpler solution of periodic pumping in a homogeneous unconfined aquifer, extending that of *Dagan* [1967] and *Neuman* [1972]. Relevant existing articles are those of *Black and Kipp* [1981] and *Rasmussen et al*. [2003], yet these deal only with confined aquifers. A general approach to time-dependent pumping, of a numerical nature, was proposed recently by *Mishra et al*. [2013]. The aim of the present note is to take advantage of the simplifications which can be achieved for periodic pumping, to derive a semianalytical solution for an unconfined homogeneous aquifer, to achieve a few aims. First, the solution can be used as a

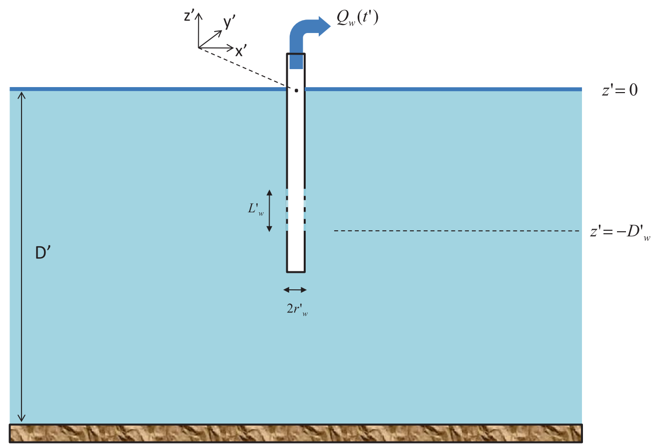

Figure 1. Definition sketch of a partially penetrating well in an unconfined aquifer as formulated in section 2.

first step in order to attempt to identify effective aquifer properties by oscillatory pumping and serve as a benchmark for numerical solutions. Second, this is anyway a necessary step toward solving the more difficult problem of heterogeneous aquifers. Third and most importantly, due to its simplicity, the solution allows to grasp the distinctive features of periodic pumping toward extensions and application.

The plan of the note is as follows: the problem is stated mathematically in the next section. Subsequently, the Green Function for a semibounded domain is derived explicitly and its properties are discussed and illustrated. Extension to a partially penetrating well pumping in an aquifer of finite thickness is covered by section 4; the note is concluded by a Summary and Conclusions.

2. Mathematical Statement of the Problem

We consider an aquifer of unbounded horizontal extent in the x', y' plane and of finite depth D', with the water table at rest at $z'=0$ (see Figure 1). A partially penetrating well of radius r'_w and length L'_w lies along the z' axis at $-D'_w < z' < -D'_w + L'_w$. The aquifer is homogeneous and anisotropic of horizontal and vertical hydraulic conductivties K_h and K_v, respectively. The flow is governed by Darcy's law and the mass conservation equation resulting in

$$s'\frac{\partial H'}{\partial t'} - K_h\left(\frac{\partial^2 H'}{\partial x'^2} + \frac{\partial^2 H'}{\partial y'^2}\right) - K_v\frac{\partial^2 H'}{\partial z'^2} = 0 \tag{1}$$

where t' is time, $H'(\mathbf{x}', t')$ is the pressure head, and s' is the specific storativity. Assuming $r'_w \ll L'_w$, we model as usual the well by a sink/source line and equation (1) becomes

$$\begin{aligned} &s'\frac{\partial H'}{\partial t'} - K_h\left(\frac{\partial^2 H'}{\partial x'^2} + \frac{\partial^2 H'}{\partial y'^2}\right) - K_v\frac{\partial^2 H'}{\partial z'^2} = \\ &-Q_w(t')\left[\chi(z'+D'_w) - \chi(z'+D'_w - L'_w)\right]\delta(x')\delta(y'), \quad z' < 0 \end{aligned} \tag{2}$$

after incorporating the line source at $x'=0, y'=0, -D'_w < z' < L'_w - D'_w$, with χ the Heaviside step function, δ the Dirac operator, and $Q_w(t')$, the well discharge divided by its length L'_w. The free surface condition, assuming that the water table drop is small compared with the aquifer thickness at rest, is linearized [e.g., *Dagan*, 1964, 1966] to become

$$n\frac{\partial H'}{\partial t'} + K_v\frac{\partial H'}{\partial z'} = 0, \quad z' = 0 \tag{3}$$

where n is the effective porosity at the water table. Thus, equation (1) is satisfied now in the domain $-D' < z' < 0$, rather than below the free surface, whose approximate elevation is given by $z' = H'(x', y', 0, t')$. The aquifer bottom is assumed to be impervious and the head far from the well is constant, i.e.,

$$\frac{\partial H'}{\partial z'} = 0, z' = -D'; H' \to 0, R' \to \infty \tag{4}$$

where $R' = (x'^2 + y')^{1/2}$. The well discharge is a periodic function of time which for simplicity is taken as harmonic. It is convenient to represent the discharge/length as the real part of a complex variable $Q_w(t') = |Q_w|\ Re[\exp(i\omega' t')]$, where $|Q_w|$ is positive and constant and ω' is the frequency. After a sufficient long time from the inception of the pumping, the head becomes also harmonic and we consider here this regime solely (a discussion of the transient regime is given by *Bakhos et al*. [2014]). We rewrite now the flow equations in terms of dimensionless variables defined by

$$x=\left(\frac{K_v}{K_h}\right)^{1/2}\frac{x'}{L},\quad y=\left(\frac{K_v}{K_h}\right)^{1/2}\frac{y'}{L},\quad R=\left(\frac{K_v}{K_h}\right)^{1/2}\frac{R'}{L},\quad z=\frac{z'}{L},\quad H=H'\frac{K_h}{|Q_w|}e^{-i\omega' t'},$$
$$s=s'L,\quad \omega=\omega'\frac{L}{K_v},\quad D_w=\frac{D'_W}{L},\quad D=\frac{D'}{L},\quad L_w=\frac{L'_w}{L},\quad t=t'\frac{K_v}{L} \tag{5}$$

where L is a reference length, unspecified at present. Substituting $Q_w(t')=|Q_w|\,\exp(i\omega t)$ and (5) in (2–4), we arrive at the final set of complex equations

$$i\omega sH-\nabla^2 H=-[\chi(z+D_w)-\chi(z+L_w-D_w)]\delta(x)\,\delta(y),\quad z<0 \tag{6}$$

$$i\omega nH+\frac{\partial H}{\partial z}=0\,,\ z=0 \tag{7}$$

$$\frac{\partial H}{\partial z}=0, z=-D; H\to 0,\quad R\to\infty \tag{8}$$

remembering that $H(R, t)$ is the complex phaser and not the actual head $H'=Re[H\ |Q_w|e^{i\omega t}/K_h](5)$.

Among the various approximations implied by (6,7) the most stringent, one is the constancy of n. Indeed, due to the presence of the unsaturated zone, there is a delayed yield mainly during the drainage period, a topic of continuing research [e.g., see *Moench*, 1997; *Mathias and Butler*, 2006; *Tartakovsky and Neuman*, 2007; *Mao et al*., 2011; *Mishra and Neuman*, 2011, for the case of constant discharge]. In spite of the advances in the field, the topic is still of debate and the results are not yet widely implemented in applications. The subject was not yet investigated for oscillatory pumping for which alternating periods of drainage and imbibition take place in the unsaturated zone. At present, a reasonable approximation is to assume a constant effective n, say averaged over the period, which is nevertheless a function of ω. We plan to investigate this topic in the future, but in the present note we assume that n is constant in space and time, yet possibly ω dependent. Indeed, it is expectable that for high ω the water table barely moves and behaves as a boundary of constant head, whereas for low ω the effective porosity tends to its usual constant value pertaining to instantaneous drainage.

In order to solve the Helmholtz equation (6) for $H(R, z)$ with boundary conditions (7,8), we take advantage of the linearity to proceed with deriving first the Green function in a semibounded domain, to be used in section 4 for the formulation of the general solution.

3. The Green Function (GF) for a Semibounded Domain

3.1. General Solution

The dimensionless GF $G(R, z, z_0)$ represents a pulsating source of unit strength in the domain $z<0$, located at $R=0$, $z=z_0$. It satisfies

$$i\omega sG-\nabla^2 G=-\delta(x)\delta(y)\delta(z-z_0),\quad z<0 \tag{9}$$

$$i\omega nG+\frac{\partial G}{\partial z}=0,\ z=0 \tag{10}$$

The dimensionless complex head field $H(R, z)$ for an aquifer of finite depth and a well of finite length is obtained by a linear operation upon G, as shown in section 4 in the sequel. Thus, the derivation of G is the key to the solution of the oscillatory well problem.

The complex solution fulfilling (9) in an unbounded domain $|z|<\infty$ is given in an analytical form by [*Carslaw and Jaeger*, 1959] (equation (10.4.12)) as follows

$$G_\infty(R, z-z_0)=-\frac{1}{4\pi r}e^{-\gamma r},\quad \gamma=(i+1)\alpha\ \text{with}\ \alpha=(\omega s/2)^{1/2}\text{i.e.}$$
$$G_{\infty R}=-\frac{1}{4\pi r}e^{-\alpha r}\cos(\alpha r)\,,\ G_{\infty I}=\frac{1}{4\pi r}e^{-\alpha r}\sin(\alpha r)\,,\ |G_\infty|=\frac{1}{4\pi r}e^{-\alpha r} \tag{11}$$

where $r=\sqrt{R^2+(z-z_0)^2}$ and the last three expressions are the real part, imaginary part and modulus of G, respectively. It is convenient to represent the solution of (9,10) [see e.g., *Dagan*, 1964] as

$$G=G_\infty(R,z-z_0)-G_\infty(R,z+z_0)+g(R,z,z_0) \tag{12}$$

where the first two terms represent the solution for a periodic point source near a boundary of constant head. By substitution in (9,10), the problem reduces to determining the regular function g in $z<0$ which satisfies

$$i\omega sg-\nabla^2 g=0,\quad z<0 \tag{13}$$

$$i\omega ng+\frac{\partial g}{\partial z}=-2\frac{\partial G_\infty}{\partial z},\quad z=0 \tag{14}$$

$$g\rightarrow 0,\quad r\rightarrow\infty \tag{15}$$

To derive g, we apply a Fourier-Bessel transform in the horizontal plane $\hat{F}(k)=(2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(R)\exp(i\mathbf{k}\cdot\mathbf{R})\ dR_x dR_y=\int_0^\infty F(R)J_0(kR)RdR$ to equations (13)–(15), which become

$$(i\omega s+k^2)\hat{g}-\frac{d^2\hat{g}}{dz^2}=0,\quad z<0 \tag{16}$$

$$i\omega n\hat{g}+\frac{d\hat{g}}{dz}=-2\frac{d\hat{G}_\infty}{dz},\quad z=0$$
$$\hat{g}\rightarrow 0,\quad r\rightarrow\infty \tag{17}$$

In (14) $\hat{G}_\infty=\int_0^\infty G_\infty(R,z_0)J_0(kR)RdR=-(4\pi)^{-1}\int_0^\infty(R^2+z_0^2)^{-1/2}\exp[-\gamma(R^2+z_0^2)^{1/2}]J_0(kR)RdR=-\exp(-\beta|z-z_0|)/(4\pi\beta)$, with $\beta=\sqrt{i\omega s+k^2}=\sqrt{2\gamma^2+k^2}$ [*Prudnikov et al.*, 1990, p. 189]. Solving the linear ordinary differential equation (16) with boundary condition (14) results in the solution for $\hat{g}$

$$\hat{g}=-\frac{1}{2\pi(i\omega n+\beta)}\exp[\beta(z+z_0)] \tag{18}$$

Inverting $\hat{g}$ (18) results in the final semianalytical expression

$$g(R,z+z_0)=-\frac{1}{2\pi}\int_0^\infty\frac{\exp[\beta(z+z_0)]}{i\omega n+\beta}kJ_0(kR)\ dk \tag{19}$$

and the solution for G is therefore given by (12), (11), and (19). The real and imaginary parts of β are given by $\beta_R=[(|\beta|^2+k^2)/2]^{1/2},\beta_I=[(|\beta|^2-k^2)/2]^{1/2},|\beta|=(k^4+\omega^2s^2)^{1/4}$ which after substitution in (19) renders g_R, g_I, and $|g|$. However, the expressions of G (12) can be simplified further as follows.

The first component of (12), $G_\infty(R,z-z_0)$ represents a pulsating source at $R=0$, $z=z_0<0$. In the immediate neighborhood of the singularity or for low frequency, for which $\gamma r\rightarrow 0$, it simplifies to the steady state expression $G_{\infty st}=-(4\pi r)^{-1}, r=[R^2+(z-z_0)^2]^{1/2}$ which is real.

The second term of (12), $G_\infty(R,z+z_0)$ stands for a pulsating source at $R=0, z=-z_0$ reflection of the first source across the unperturbed water table and it is regular in the flow domain $z<0$.

Finally, the term g (19) encapsulates the impact of the water table via boundary condition (14). While it had a simple analytical expression for a sudden change of the discharge [e.g., *Dagan and Lessof*, 2011], it has to be evaluated by a quadrature for an oscillatory one. We found that for the large range of values of the parameters ω, n, s of Figure 2, covering most conceivable applications, it can be approximated by its expression in an incompressible medium, i.e., $s=0$, for which $\beta=k$. Hence, (19) simplifies to

$$g(R,z+z_0)=-\frac{1}{2\pi}\int_0^\infty\frac{\exp[k(z+z_0)]}{i\omega n+k}\quad kJ_0(kR)\ dk \tag{20}$$

which is adopted in (12) as a valid approximation of G in the rest of this work. The simplification of (19) achieved by (20) is in the separation of the real and imaginary parts of G, which now become

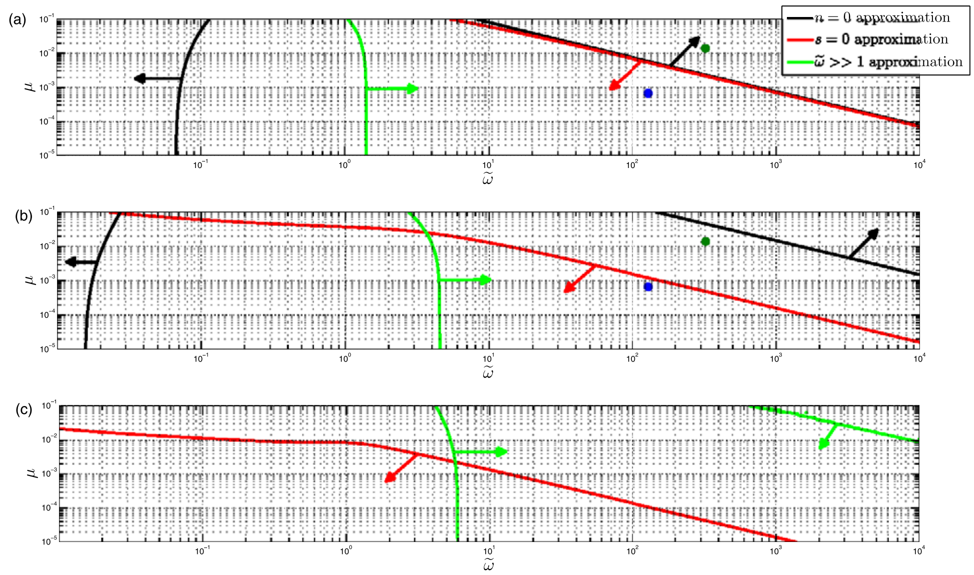

Figure 2. Zones of validity of various approximations of the Green Function amplitude in the $\mu, \tilde{\omega}$ plane for $z=-1$ and three values of the dimensionless R: (a) R=0.2, (b) R=1, and (c) R=5. Arrows indicate the zone for which the error is less than 3%. The points pertain to the $\mu, \tilde{\omega}$ extreme values of the experiment at the Boise Hydrogeophysical Research Site.

$$
\begin{aligned}
G_R(R,z,z_0) &= -\frac{1}{4\pi r}\cos(\alpha r)|_{r=[R^2+(z-z_0)^2]^{1/2}} + \frac{1}{4\pi r}\cos(\alpha r)|_{r=[R^2+(z+z_0)^2]^{1/2}} \\
&\quad -\frac{1}{2\pi}\int_0^\infty \frac{k}{k^2+\omega^2 n^2}\; e^{k(z+z_0)}\; kJ_0(kR)dk \\
G_I(R,z,z_0) &= \frac{1}{4\pi r}\sin(\alpha r)|_{r=[R^2+(z-z_0)^2]^{1/2}} - \frac{1}{4\pi r}\sin(\alpha r)|_{r=[R^2+(z+z_0)^2]^{1/2}} \\
&\quad +\frac{\omega n}{2\pi}\int_0^\infty \frac{1}{k^2+\omega^2 n^2}\; e^{k(z+z_0)}\; kJ_0(kR)dk
\end{aligned}
\qquad (21)
$$

where $\alpha=(\omega s/2)^{1/2}$ while $|G|=[G_R^2+G_I^2]^{1/2}$, $phase(G)=\theta=\tan^{-1}(G_I/G_R)$.

The two asymptotic simple limits of g and G are: (i) $\omega n \ll 1$ (confined aquifer and/or low frequency) in which case $g \to -2G_\infty(r, z+z_0)$ in (19,20) whereas $G \to G_\infty(R, z-z_0)+G_\infty(R, z+z_0)$, the rigid wall (at $z=0$) solution. Since in this case it is reasonable that $\omega s \ll 1$, the impact of storativity is limited and the solution tends to be the simple expression obtained by using $G_{\infty st}=-1/(4\pi r)$ (rigid medium). (ii) $\omega n \gg 1$ (unconfined aquifer and/or high frequency) for which in (12) $g \to 0$ and $G \to G_\infty(R, z-z_0)-G_\infty(R, z+z_0)$, i.e., the boundary $z=0$ is of constant head $G=0$, as if the aquifer is bounded from above by a reservoir of very large conductivity, of standing fluid.

The intermediate cases of $\omega n=\mathbf{0}(1)$ requires the incorporation of g (20) in the solution, and its derivation is one of the main contribution of this study.

3.2. Illustration and Discussion

The Green Function is the building block of the complete solution, as shown in the next section, and it is instructive to illustrate its properties. Furthermore, by (5) $H'(R', z', t')=(|Q_w|/K_h)Re[G\; e^{i\omega' t'}]$ represents the head field for a partially penetrating well close to the water table, for sufficiently large distances (relative to L_w') from the well. Hence, the solution can be readily applied to such cases.

Since for a semibounded domain, the only length scale of the problem is $z_0' < 0$, the depth of the source, we take without loss of generality $L=-z_0'$, i.e., $z_0=-1$. Then, it is seen that G (21) depends on $R, z\pm1$ and the two parameter ωs and ωn. For the sake of the analysis, we prefer to select $\mu=s/n=s'|z_0|/n$ and $\tilde{\omega}=\omega n =\omega'|z_0'|n/K_v$ as the two independent governing parameters.

While the general solution (12) and either (19) or (20) can be readily used in applications, it simplifies even more for different regimes associated with ranges of values of the parameters $\tilde{\omega}$ and μ. To illustrate them,

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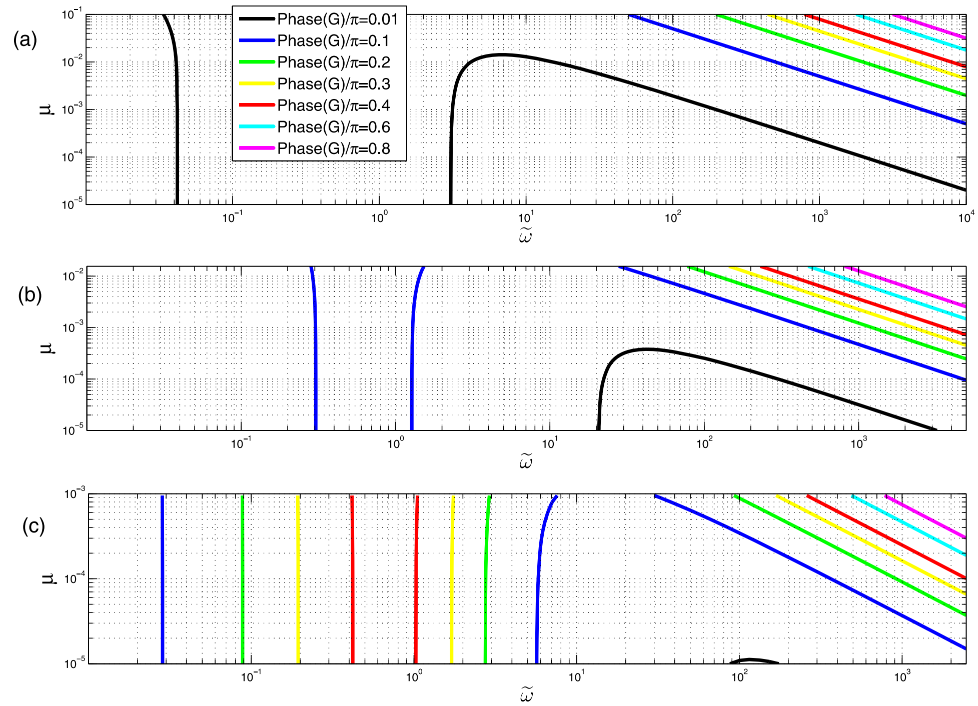

Figure 3. The Green Function phase at $z=-1$ as a function of μ and $\tilde{\omega}$ for three values of the dimensionless radius *R*: (a) R=0.2, (b) R=1, and (c) R=5.

we represent in Figure 2 the zones of validity of the various approximations for the amplitude $|G|$ at $z=-1$ (i.e., at the depth of the source, of maximal $|G|$), for the three representative values of $R=0.2, 1, 5$.

A first finding is that the approximation $\tilde{\omega} \gg 1$ (to the right of the green curve) pertaining to sufficiently high frequency for which $g=0$ in equation (12), i.e., constant head at $z=0$, applies to $\tilde{\omega} > 1, 4, 6$ for Figures 2a–2c, respectively. This covers many conceivable values in applications. In contrast, for very low values of $\tilde{\omega}$ and for $R=0.2$,1 the approximation $n=0$ (left of black curve, confined aquifer) may apply. Combined with the approximation $s=0$, which is valid in this zone, *G* reduces to the simple expression of a source in a rigid and confined aquifer. This is consistent with the solution for constant rate continued pumping at large times. There is an intermediate zone of low $\tilde{\omega}$ values in which *g* has to be incorporated in the expression of *G*.

Similarly, the approximation $s=0$ manifesting in the expressions of G_∞, which tends to $-(4\pi r)^{-1}$ applies to a large domain of values of μ and $\tilde{\omega}$. In contrast, storativity has to be accounted for combinations of high frequency and low water table specific yield *n*. The latter may be related to the impact of the unsaturated zone which serves as a buffer and diminishes *n* at high frequency. In the border zones of the upper-right corners of Figures 2a and 2b, both approximations $\tilde{\omega} \to \infty$ and $n \to 0$ apply, i.e., the solution degenerates into $G=G_\infty(R, z-z_0)$ (11), pertaining to a pulsating source in an infinite domain and the water table presence is inconsequential.

As mentioned in the introduction, hydraulic tomography by a step function discharge was conducted at the Boise Hydrogeophysical Research Site [*Cardiff et al.*, 2013a]. An oscillatory tomography test is envisaged in the near future and a few various parameters were provided to the authors (W. Barrash, personal communication, 2014). For the sake of illustration, we have plotted in Figure 2 the two extreme combinations of parameters values, though *R* were somewhat different from those of the figure.

Though the complete expression of *G* (12) is not complicated, the discussion of the validity of various approximations carried out so far is relevant to the identification process, as it points out to regimes and zones in which the oscillatory pumping test is not helpful in determining some parameters. The discussion is also of interest in elucidating the impact of the water table.

In a similar manner, the phase θ of *G* (12) relative to that of the discharge, is represented in Figure 3 for the range of the more common values $10^{-5} < \mu < 10^{-3}$, for $z=-1$, $R=0.2, 1, 5$. In Figure 3a and 3b for which the approximation $s=0$ applies in most of the area, the phase is of low value relative to π. It is only for the less common values of large $\tilde{\omega}$ and large μ (for which s plays a role) that the phase becomes sizable. The phase may be larger for $R=5$, but the amplitude is quite low there.

4. Impact of Finite Depth and Finite Well Length

Starting with the impact of the finite depth D, the solution for the Green function can be obtained by solving again equations (16) and (17) for $\hat{g}$, but after replacing (17) by $d\hat{G}/dz=0$ for $z=-D$. The solution of the linear differential equation is easily obtained as

$$\hat{g}=-\frac{e^{\beta(z+z_0)}}{2\pi A}+\frac{e^{-2\beta D}}{2\pi}\frac{Be^{\beta z}-Ae^{-\beta z}}{A+Be^{-2\beta D}}\left[\frac{e^{\beta z_0}}{A}-\frac{\sinh(\beta z_0)}{\beta}\right] \text{ where} \quad A=i\tilde{\omega}+\beta,\ B=i\tilde{\omega}-\beta,\ \tilde{\omega}=\omega n \tag{22}$$

which indeed tends to (18) for $D\to\infty$. It is emphasized that in line with the simplification of (19) β should be replaced by k in most conceivable applications.

The inversion of $\hat{g}$ requires a quadrature over $J_0(kR)$ (see 20) which can be simplified by carrying out first an expansion of (22) in $\varepsilon=(B/A)\exp(-2\beta D)$ for $D\gg 1$. Thus

$$\hat{g}=-\frac{e^{\beta(z+z_0)}}{2\pi A}+\frac{e^{-2\beta D}}{2\pi}\frac{Be^{\beta z}-Ae^{-\beta z}}{A}\left[\frac{e^{\beta z_0}}{A}-\frac{\sinh(\beta z_0)}{\beta}\right]\sum_{m=0}^{\infty}(-1)^m\varepsilon^m \tag{23}$$

The series in (23) can be viewed as an array of singularities at $z=\pm z_0\pm 2mD$, similarly to the case of constant recharge (*Dagan and Lessof* [2011]; Figure 2). The possible advantage of (23) upon (22) is the quick decay of the series and the possibility to use a few terms only. Thus, the leading order terms of expansion (23), for $m=0$ and singular at $z=\pm z_0-2D$, are given by

$$\hat{g}\to-\frac{e^{\beta(z+z_0)}}{2\pi A}-\frac{e^{-2\beta D}}{2\pi}\left[\frac{e^{-\beta(z-z_0)}}{A}-\frac{e^{-\beta z}\sinh(\beta z_0)}{\beta}\right] \tag{24}$$

which leads to the Green Function (12) for infinite depth supplemented by the reflection of the three singularities present in (12) across $z=-D$.

The next generalization is for a well of finite length extending over $-D_w'<z_0'<-D_w'+L_w'$. Taking for the reference length $L=L_w'$, the pumping well length, we integrate first the water table contribution (22) to the Green function to obtain the general expression

$$\bar{\hat{g}}=\int_{-D_w}^{-D_w+1}\hat{g}(z,z_0)\,dz_0=-\frac{e^{\beta z}(e^{-D_w+1}-e^{-D_w})}{2\pi\beta A}+\frac{e^{-2\beta D}}{2\pi\beta}\frac{Be^{\beta z}-Ae^{-\beta z}}{A+Be^{-2\beta D}} \times\left\{\frac{e^{-D_w+1}-e^{-D_w}}{A}-\frac{\cosh[\beta(D_w-1)]}{\beta}+\frac{\cosh(\beta D_w)}{\beta}\right\} \tag{25}$$

where it is reminded that $D_w=D_w'/L_w'>1$ (Figure 1). Again g is obtained by inversion of the Fourier Bessel transform as in (20), with the simplification $\beta=k$ in most applications.

As for the Green Function in an unbounded domain (11), the integral of $G_\infty(r)$, with $r=[R^2+(z-z_0)^2]^{1/2}$, $\gamma=(i+1)(\omega s/2)^{1/2}$ is given by

$$\bar{G}_\infty(R,z)=-\frac{1}{4\pi}\int_{-D_w}^{-D_w+1}\frac{e^{-\gamma r}}{r}\,dz_0 \tag{26}$$

and it does not have an analytical expression and has to be carried out numerically.

It is seen that the computation of the various terms of G (12) needs a quadrature. However, analytical expressions are available for a few limit cases of wide applications, as shown in the previous section. Thus, for $s=0$ $(\gamma=0)$, we get in (26) the well-known steady state expression

$$\bar{G}^{st}_{\infty}(R,z)=-\frac{1}{4\pi}\int_{-D_w}^{-D_w+1}\frac{1}{r}\,dz_0=-\frac{1}{4\pi}\ln\frac{z+D_w-1-\left[(z+D_w-1)^2+R^2\right]^{1/2}}{z+D_w-\left[(z+D_w)^2+R^2\right]^{1/2}} \quad (27)$$

Finally, the head, solution of (2,3,4) is determined by (5) as follows

$$H^{'}(R^{'},z^{'},t^{'})=Re\left[\frac{|Q_w|}{K_h}\bar{G}(R,z)e^{i\omega t}\right] \quad (28)$$

where the variables $R^{'},z^{'},t^{'}$ are related to the dimensionless ones by (5).

The application of the results to pumping tests is achieved by using different intervals along the pumping well, varying the discharge magnitude and frequency, and measuring the oscillatory head at various points along observations wells. Hydraulic tomography aims at identifying the spatial variation of K in the aquifer volume surrounding the pumping well (see references in Introduction). This has to be achieved by using the head signal and the solution of the flow problem in an inverse mode. In the similar problem pertaining to a stepwise discharge [*Cardiff et al*., 2013a], this has been achieved by a fully numerical approach which employs complex codes and requires significant computational power.

The present study regards the aquifer as homogeneous and it is suggested that the simple solution presented here can be used as a first step toward identifying parameter average values, which may serve as a starting point for the tomographical investigation. Furthermore, the deviations of the measured heads from the computed ones are indicative of the degree of heterogeneity of the aquifer. Toward this aim, we remind that the modulus and the phase of the solution $H^{'}(R^{'},z^{'},t^{'})$ (28) depends on the following parameters (see 5): Q_w/K_h, K_v/K_h, $\omega^{'}L_w^{'2}s^{'}/K_v$, $\omega^{'}L_w^{'}n/K_v$, $D_w^{'}/L_w^{'}$ and $D^{'}/L_w^{'}$. Among these parameters Q_w, $\omega^{'}$, $L_w^{'}$, $D_w^{'}$ and $D^{'}$ are given input ones whereas K_v, K_h, $s^{'}$, and n have to be identified with the aid of the measured $|H^{'}|$ and *phase*($H^{'}$) at different points of coordinates $R^{'},z^{'}$.

5. Summary and Conclusions

The paper derives an analytical solution of the head field in a homogeneous and anisotropic unconfined aquifer, pumped periodically by a partially penetrating well. It is assumed that pumping takes place a sufficiently long time to render the solution periodic. The simplifying assumptions are the ones usually adopted in the literature for the common constant discharge test: the well radius is much smaller than its length, the free surface condition is linearized, anisotropy principal axes are horizontal and vertical, specific storativity s and water table effective porosity n are constant. Periodic pumping provides an additional degree of freedom by varying the frequency ω, which allows for regarding n as frequency dependent.

An analysis of the dependence of the head amplitude and phase on the various parameters of the problem permits to delineate ranges of values for which the solution at a given point simplifies to those pertaining to a rigid aquifer ($s=0$), to a constant head boundary or a rigid wall ($n=0$) upper boundary condition. Generally, the water table has a large impact on the head making the present solution a useful one in applications. Furthermore, it is found that for a large range of parameter values the assumption of constant head boundary, rather than an impervious one, is applicable. The storativity has generally a smaller impact except for high frequency and for zones close to the well and the bottom. The phase of the head relative to that of the pumping is significant in the neighborhood of the water table or in zones in which s is dominant.

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References

Bakhos, T., M. Cardiff, W. Barrash, and P. Kitanidis (2014), Data processing for ocscillatory pumping tests, *J. Hydrol.*, *511*, 310–319.
Black, J., and K. Kipp Jr. (1981), Determination of hydrogeological parameters using sinusoidal pressure tests: A theoretical appraisal, *Water Resour. Res.*, *17*, 686–692.
Cardiff, M., and W. Barrash (2011), 3-d transient hydraulic tomography in unconfined aquifers with fast drainage response, *Water Resour. Res.*, *47*, W12518, doi:10.1029/2010WR010367.
Cardiff, M., W. Barrash, and P. Kitanidis (2012), A field proof-of-concept of aquifer imaging using 3-d transient hydraulic tomography with modular, temporarily-emplaced equipment, *Water Resour. Res.*, *48*, W05531, doi:10.1029/2011WR011704.
Cardiff, M., W. Barrash, and P. Kitanidis (2013a), Hydraulic conductivity imaging from 3-d transient hydraulic tomography at several pumping/observation densities, *Water Resour. Res.*, *49*, 7311–7326, doi:10.1002/wrcr.20519.
Cardiff, M., T. Bakhos, P. Kitanidis, and W. Barrash (2013b), Aquifer heterogeneity characterization with oscillatory pumping: Sensitivity analysis and imaging potential, *Water Resour. Res.*, *49*, 5395–5410, doi:10.1002/wrcr.20356.
Carslaw, H., and J. Jaeger (1959), *Conduction of Heat in Solids*, p. 510, 2nd ed., Oxford Univ. Press, London.

Dagan, G. (1964), Second order linearized theory of free surface flow in porous media, *La Houille Blanche*, *8*, 901–910.
Dagan, G. (1966), The solution of the linearized equations of free-surface flow in porous media, *J. Mecanique*, *5*(2), 207–215.
Dagan, G. (1967), A method of determining the permeability and effective porosity of unconfined anisotropic aquifers, *Water Resour. Res.*, *3*, 1059–1071.
Dagan, G., and S. Lessof (2011), Flow to partially penetrating wells in unconfined heterogeneous aquifers: Mean head and interpretation of pumping tests, *Water Resour. Res.*, *47*, W06520, doi:10.1029/2010WR010370.
Kruseman, G., and N. Ridder (1990), *Analysis and Evaluation of Pumping Test Data*, ILRI Publ. 377 pp., Wageningen, Netherlands.
Mao, D., L. Wan, T. C. J. Yeh, C. H. Lee, K. C. Hsu, J. C. Wen, and W. Lu (2011), A revisit of drawdown behavior during pumping in unconfined aquifers, *Water Resour. Res.*, *47*, W05502, doi:10.1029/2010WR009326.
Mathias, S., and A. Butler (2006), Linearized Richards' equation approach to pumping test analysis in compressible aquifers, *Water Resour. Res.*, *42*, W06408, doi:10.1029/2005WR004680.
Mishra, P., and S. Neuman (2011), Saturated-unsaturated flow to a well with storage in a compressible unconfined aquifer, *Water Resour. Res.*, *47*, W05553, doi:10.1029/2010WR010177.
Mishra, P., V. Vessilinov, and H. Gupta (2013), On simulation and analysis of variable-rate pumping tests, *Ground Water*, *51*, 469–473.
Moench, A. (1997), Flow to a well of finite diameter in a homogeneous, anisotropic water table aquifer, *Water Resour. Res.*, *33*, 1397–1407.
Neuman, S. (1972), Theory of flow in unconfined aquifers considering delayed response of the water table, *Water Resour. Res.*, *8*, 1031–1045.
Prudnikov, A., Y. Brychkov, and O. Marichev (1990), *Integrals and Series*, vol. 2, Gordon and Breach, N. Y.
Rasmussen, T., K. Haborak, and M. Young (2003), Estimating aquifer hydraulic properties using sinusoidal pumping at the savannah river site, south Carolina, USA, *Hydrol. J.*, *11*, 466–482.
Tartakovsky, G., and S. Neuman (2007), Three-dimensional saturated-unsaturated flow with axial symmetry to a partially penetrating well in a compressible unconfined aquifer, *Water Resour. Res.*, *43*, W01410, doi:10.1029/2006WR005153.

