Understanding Concepts in Mathematics Rubric Manual

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Understanding Concepts in Mathematics Rubric

Overview:

Conceptual understanding in mathematics refers to understanding and mental representations of mathematics principles and the interrelations between pieces of knowledge in various domains of mathematics (Rittle-Johnson & Alibali, 1999; Rittle-Johnson, Fyfe,& Loehr, 2016). It has also been described as “an integrated and functional grasp of mathematical ideas” (Kilpatrick et al., 2001, p118). For example, the concept of addition involves understanding that combining similar groups forms a new group (usually larger). The concept of subtraction involves taking parts away of a quantity or comparing two similar things to find the difference in quantity. Conceptual understanding relates to the meaning of the numbers, relationships, and symbols.

Understanding of concepts and knowledge of procedures are both important; they work together with one type of knowledge supporting the other (Baroody, Feil, & Johnson, 2007; Rittle-Johnson, Schneider, & Star, 2015). When students are taught in a manner that develops conceptual understanding in addition to procedural skills, they show greater achievement and retention than those who receive instruction focused on procedures alone (Geary, et al., 2008; Rittle-Johnson, Schneider, & Star, 2015). Conceptual knowledge is more efficient than rote memorization because it applies across many novel situations. It increases the likelihood that students will apply what they’ve learned in new situations, both in and out of school (Rittle-Johnson & Alibali, 1999).

The Understanding Concepts rubric was designed for use by supervisors and administrators to reliably evaluate teachers’ implementation of practices that support the development of students’ conceptual understanding. The rubric provides specific, accurate, and actionable feedback to special education teachers about the quality of their instruction, and ultimately, improve the outcomes for students with disabilities. The purpose of this manual is to provide technical information for implementing the Understanding Concepts Rubric rubric as a tool for evaluation and feedback.

This rubric includes 5 components. These are:
1) Content of Instruction
2) Design of Instruction

3) Delivery of Instruction
4) Student Engagement
5) Providing Feedback

Under these 5 components, there are 14 items. For each item, there are five levels of implementation. Observing either live or from video, the observer assigns a rating based on a scale that ranges from **Implemented** to **Not Implemented.** The rater selects one score from among the following choices:
3--Implemented, meaning the teacher's performance aligns with the descriptor,
2+
2--Partially Implemented, meaning the teacher's instruction reflects this item but there are flaws or missing components in the way in which it is implemented,
2-
1--Not Implemented, meaning the item is either implemented poorly or should have been observed but is not,
and **Not Applicable (N/A)**, a category that recognizes that given the lesson context and what is taught in previous lessons, not every item will be observed across every observation.

Preparation for the Observation

There are several materials you will need in order to use the Understanding Concepts Rubric to conduct the observations. First, you should ensure you have everything you need to conduct the observation including pencils, a clipboard (or something hard to write on), and a copy of the Understanding Concepts Rubric. The Understanding Concepts Rubric is your scoring form and your note-taking space. Use the margins and the backs of pages to write notes of the things you observe that help you determine what ratings to assign. The notes will be also useful when you need to provide feedback to the teacher.

Understanding the Understanding Concepts Rubric Structure

There are 14 items in this rubric. Each item is listed in a table below with an explanation and description of the intention of the item to help clarify its meaning. Each item has five levels of implementation. Descriptors are given for high, middle, and low levels of implementation. Examples are included to help you interpret the meaning of the different implementation levels. You should consider these descriptions and examples as you determine the implementation level for each item.

Assigning Rating on Understanding Concepts Rubric

The Understanding Concepts Rubric rating scale includes a score of: **1)"Not Implemented," 2) "Partially Implemented," and 3) "Implemented,"**. The "Partially Implemented" category is further divided to allow for assigning a **2-, a 2, or a 2+**, to indicate the degree to which the item is partially implemented. A **2-** indicates a very low level of partial implementation, whereas a **2+** can be used in cases where the item is almost fully implemented but not quite.

Observing either live or from video, you assign a rating on the basis of the observations. Assign a rating that comes closest to describing the observation even if not an exact match. For each item, assign a single rating, unless it is N/A.

Because the duration of a class may be 40 minutes or more, it is helpful to note whatever is observed, even at a low level. Then if a higher level item implementation makes the previous item inaccurate, the previous choice can simply be changed. This is especially useful when some items need to be observed throughout the whole lesson. For example, after observing the teacher "allows adequate time for students to think or respond," the observer should check 'Partially Implemented', but if the teacher continues to allow adequate time for students to think or respond in that way until the end of the lesson, 'Partially Implemented' should be crossed out and a higher level of item implementation is checked.

Description of the Understanding Concepts Rubric rubric

Component 1: Content of Instruction

The purpose of this section is to describe aspects of mathematics lesson content that research has identified as effective in developing conceptual understanding in mathematics.

Elements of Component 1 are:

Item 1- The lesson is consistently focused on conceptual understanding of critical mathematics concepts (amount appropriate for meaningful development).

Item 2-The teacher uses visual representations that support conceptual understanding, i.e., show the appropriate size relations and map to the meaning of number when in context.

Component 2: Design of Instruction

This component contains items that describe the way a lesson is structured. Rather than focusing on the content of the lesson, this is focused on how a teacher has planned to systematically present the material while making appropriate connections to previously learned material, elements within the lesson, and students' knowledge.

Elements of Component 2 are:

Item 3-To begin instruction, the teacher selects a simple or familiar context or representation that effectively provides meaning for the numbers.

Item 4-The teacher effectively reviews or teaches key vocabulary and/or symbols.

Item 5-There is an explicit systematic progression within and/or across lessons, e.g., a step-by-step presentation, a graduated sequence of representations, or explicit connections to previous lessons.

Item 6-The teacher engages students in making connections related to the concept(s). For example, connections can be made between:
· concepts and students' background knowledge,
· contexts and representations,
· informal and formal representations.

Component 3: Delivery of Instruction

This component contains items that describe the manner in which the teacher delivers the instruction. This includes the ways in which the teacher is responsive to students' needs and the quality of the teacher's communication.

Elements of Component 3 are:

Item 7-The teacher presents a range of examples that is responsive to the needs of the students.

Item 8-The teacher consistently discusses mathematical ideas with language that is clear, accurate, and precise.

Item 9-The teacher clearly and sufficiently verbalizes and models reasoning (i.e., think-aloud).

Component 4: Student Engagement

This component contains items that describe how the teacher has planned for and implements opportunities for students to be engaged in the lesson and practice working with the concept.

Elements of Component 4 are:

Item 10-The teacher provides students with sufficient opportunity to verbalize their understanding and/or reasoning.

Item 11-The teacher encourages students to use mathematical vocabulary and/or symbols throughout the lesson.

Item 12-The teacher provides students with practice adequate to supporting the development of conceptual understanding.

Component 5: Providing Feedback

This component contains items that describe the nature of the feedback provided to students.

Elements of Component 5 are:

Item 13-The teacher uses questions and prompts related to visual representations to encourage students to monitor and correct their work.

Item 14-Feedback is consistently linked to mathematical reasoning and concepts.

Psychometric Properties

The Understanding Concepts Rubric rubric has been developed through a rigorous process to ensure that it is a valid and reliable instrument. Each item included within the rubric comes from an analysis of the existing research establishing these instructional practices as effective for students with high incidence disabilities. Additionally, the rubric has been reviewed by content experts in the field to support content validity.

Further psychometric review is ongoing.

Key Terms on the Understanding Concepts Rubric

“**Consistently**” means every time the opportunity arises, the teacher responds in the same or an appropriately similar way. It is different from continuously.
“**Effectively**” means adequate to accomplish a purpose or produce intended or expected results.
“**Adequate**” means as much or as good or as necessary to accomplish a purpose or produce intended or expected results.
“**Frequently**” means regularly or often.
“Visual representations” refer to concrete and pictorial representations of numbers, concepts, and relationships. Pictorial representations can be informal or formal. For the purposes of this rubric, visual representations include, but are not limited to, concrete manipulatives, concrete or pictorial bar models, tape diagrams, pictures, icons, number lines, graphs, etc.

Item Descriptions and Examples

Item 1: The lesson is consistently focused on conceptual understanding of critical math concepts (amount appropriate for meaningful development). This item addresses the need for students to have access to the key concepts and foundational proficiencies that will support their success with the general education curriculum in current and future years. Critical math concepts are those that help student reach the most critical standards or big ideas. In grades kindergarten - 5, these critical concepts involve counting concepts, properties of whole numbers and operations, number composition and decomposition, and meaning of base-10 (including but not limited to place value), and reasoning underlying the algorithms. In grades 4 -8, these concepts extend to understanding the meaning of fractions, decimals, ratios, and percents, and solving problems with numbers in these forms. The lesson may include one or more of these concepts. If more than one concept is addressed or practiced, there should not be so many that the students are confused or struggling to understand them all.		
Implemented-3	**Partially Implemented-2**	**Not Implemented-1**
The lesson is consistently focused on conceptual understanding of critical mathematics concepts (amount appropriate for meaningful development).	The lesson addresses conceptual understanding of critical concepts but not consistently, OR the lesson addresses too many critical concepts for meaningful development.	The lesson does not address conceptual understanding, OR the lesson does not address critical concepts.
Examples: ● While creating a bar graph, the teacher emphasizes the relation between the size of the bar and the quantity it represents. The teacher leads the discussion to generalizable statements about the length of bars and the quantities they represent. ● The lesson maintains focus on the meaning of multiplying	Examples: ● The lesson is focused on on representing quantities on a bar graph, but the teacher digresses into describing different types of graphs. As a result, the teacher does not emphasize the relation between sizes of the bar and the quantities they represents. ● The teacher allows a student comment to lead to a long discussion of another unrelated	Examples: ● The lesson is focused on on creating a bar graph without emphasizing counting concepts, number composition, or other important concepts. ● The lesson is focused on telling time with an analog clock.

by 10 and by 100. All modeling, think-aloud, and examples are related to the concept of powers of 10. ● The lesson is focused on using common units to find equivalent fractions. The teacher reviews important foundational concepts like the meaning of the denominator as appropriate but maintains focus on common units.	mathematics topic rather than integrating students' knowledge into the topic or stopping the discussion early. ● The lesson introduces two-digit by one-digit multiplication with an area model. The teacher allows and also introduces multiplication by powers of 10. ● The lesson moves quickly from the concept of multiplying by one to create an equivalent fraction to using common units to compare fractions to and on further to discussion to the meaning of the denominator. Students do not have an opportunity to practice or demonstrate understanding of each of these topics individually.	● The lesson does not move beyond an activity in which students match fraction bars of the same size and record the fraction names on a chart. While the activity is based on the concept of equivalence, the lesson is focused on a sequence of actions without explicit connection to the concept.

Item 2-The teacher uses visual representations that support conceptual understanding, i.e, show the appropriate size relations and map to the meaning of numbers when in context.

The purpose of this item is to ensure that visual representations are used in the lesson. The representations also need to be consistent with the concept that is the focus of the lesson. Visual representations should meaningfully represent the quantities that are linked to the numbers. Visual representations can include manipulatives, simple drawings, number lines, bar models, tape diagrams, arrays, graphs etc.

Implemented-3	**Partially Implemented-2**	**Not Implemented-1**
The teacher uses visual representations that support conceptual understanding, i.e., show the appropriate size relations and map to the meaning of number when in context.	The teacher uses some visual representations that support conceptual understanding, OR the teacher uses visual representations that somewhat support conceptual understanding.	The teacher does not use visual representations, OR the teacher uses visual representations that do not support conceptual understanding.
Examples: ● The teacher uses base-10 blocks and drawings of them that reflect the 10-to-1 relationship. ● The teacher focuses on a linear meaning of fractions using a context of submarine sandwiches that are represented with bar models and number lines. ● The lesson involves constructing arrays to model multiplication. The teacher clearly connects the visual model to the meaning of the operation and numbers used.	Examples: ● The teacher uses base-10 blocks, but the drawings of these blocks on the board do not reflect the 10-to-1 relationship. ● The teacher uses a combination of circular and linear representations of fractions with a context of submarine sandwiches. ● The lesson involves constructing arrays to model multiplication without engaging in the conceptual basis for using them. The connection between the array and the meaning of multiplication is left implicit.	Example: ●The teacher uses objects to represent hundreds, tens, and ones that are not proportional (do not reflect the 10 to 1 relationship). ● The teacher talks about imagining the submarine sandwich in a story problem as a number line but does not draw or use a number line. ● The teacher presents a lesson on the meaning of multiplication relying only on writing numbers on the board. There are no visual models.

Item 3-To begin instruction, the teacher selects a simple or familiar context or representation that effectively provides meaning for the numbers. This item examines how effectively the teacher connects to what students already know prior to beginning instruction. The teacher provides an initial context for the numbers or initially uses a visual representation (manipulative or model) that is simple or familiar enough for students to engage with meaningfully from the start of the lesson without the context or representation creating confusion. For example, a simple drawing can help students make sense of complex concepts such as equivalence, different meanings for each of the operations, etc.		
Implemented-3	**Partially Implemented-2**	**Not Implemented-1**
To begin instruction, the teacher selects a simple or familiar context or representation that effectively provides meaning for the numbers.	To begin instruction, the teacher selects a context or representation that does not effectively provide meaning for the numbers.	To begin instruction, the teacher does not use a context or a representation, OR the teacher selects a context or representation is confusing or inaccurate.
Examples: ● The teacher begins a lesson on modeling fractions on a number line by asking students to draw the fraction ¼ any way they would like on a piece of paper. The teacher explicitly connects these drawings to ¼ on a number line. ● To begin a lesson on the modeling multiplication with an array, the teacher uses the context of a class watching a movie to arrange chairs in rows and columns. (familiar context that provides meaning for numbers)	Examples: ● The teacher begins a lesson on modeling fractions on a number line by drawing a number line on the board. The teacher does not connect this representation to a context or more intuitive representation of fractions. ● To begin a lesson on modeling multiplication with an array, the teacher creates an elaborate story about a school with different sized classrooms that hold different numbers of desks. (unnecessarily complex, reduces effectiveness)	Examples: ● The teacher begins a lesson on understanding fractions by discussing the meaning of the numerator and denominator. The teacher does not use a context or representation. ● To begin a lesson on modeling multiplication with an array, the teacher uses a context of feeding a pet the same amount of food each day. (context better supports repeated addition and is not easily translated to an array)

<table>
<tr><td colspan="3">Item 4-The teacher effectively reviews or teaches key vocabulary and/or symbols.
It is important to ensure that the meanings of vocabulary and symbols are clear to students. Ensuring this clarity is important for students with disabilities such as memory or language processing rather than assuming they will remember or infer the meanings. This may occur at the beginning of a lesson or may occur mid-way through a lesson as appropriate. Students may provide the review if they are able to provide clear and accurate definitions and/or examples. If they cannot, the teacher should provide clear and explicit definitions. This is important for students with disabilities such as memory or language processing rather than assuming they will remember or infer the meanings. Effectively reviewing or teaching involves a clear, timely, concise, focused explanation of the term or symbol.</td></tr>
<tr><td>Implemented-3</td><td>Partially Implemented-2</td><td>Not Implemented-1</td></tr>
<tr><td>The teacher effectively reviews or teaches key vocabulary and/or symbols.</td><td>The teacher reviews or teaches key vocabulary and/or symbols but not effectively.</td><td>The teacher does not review or teach key vocabulary and/or symbols.</td></tr>
<tr><td>Examples:
● After stating they will be comparing fractions with unlike denominators, the teacher quickly verifies that students remember key vocabulary (numerator, denominator, and equivalence) by asking for a definition and providing a concise definition when students do not remember.

● After an activity in which students use cubes to model a simple context involving exponents, the teacher clearly explains the terms (base and exponent), linking the definitions to the activity, and clearly models the symbolic notation.</td><td>Examples:
● After stating they will be comparing fractions with unlike denominators, the teacher spends several minutes reviewing vocabulary words by providing hints and having student try to guess the meaning.

● During a lesson on adding with unlike denominators, the teacher reviews the terms numerator and denominator after discovering students are confused about the terms.

● After an activity in which students use cubes to model a simple context involving exponents, the teacher explains the terms (base and exponent) and the symbolic notation without reference to the activity.</td><td>Examples:
● The teacher presents a lesson on comparing fractions with unlike denominators without a review of vocabulary such as numerator, denominator, and equivalence.

● The teacher leads students through an activity modeling exponents but does not provide any definitions or explanations of terms or symbolic notation.</td></tr>
</table>

Item 5-There is an explicit systematic progression within and/or across lessons that supports conceptual understanding, e.g., a step-by-step presentation, a graduated sequence of representations, or explicit connections to previous lessons. This item assesses whether the teacher organizes instruction of the concept into a logical sequence to help students build connections between ideas. Connections to prior learning, breaking down concepts into smaller components, or providing representations that gradually become more sophisticated are forms of scaffolding. Systematic instruction can be a sequential presentation of steps, organized presentation of examples and non-examples, or an organized compare and contrast. Systematic instruction may teach students to develop representations and gradually formalize these, transitioning to abstract symbols over time. Regardless of the structure, it is logical, clear, organized, and provides unambiguous connections.		
Implemented-3	**Partially Implemented-2**	**Not Implemented-1**
There is an explicit systematic progression within and/or across lessons that supports conceptual understanding, e.g., a step-by-step presentation, a graduated sequence of representations, or explicit connections to previous lessons.	There is a somewhat explicit or somewhat systematic progression within and/or across lessons, OR there is an explicit, systematic progression within and/or across lessons that somewhat supports conceptual understanding.	There is not an explicit or systematic progression within or across lessons that supports conceptual understanding.
Examples: ● The teacher begins the lesson by connecting to the previous day's lesson using base-10 blocks to model two-digit numbers. In this lesson, students model three digit numbers with base-10 blocks. ● The lesson includes a graduated sequence of representations (concrete, representational, abstract) that are presented systematically in increasing	Examples: ● The teacher states they will build on yesterday's activity. She begins the lesson by drawing squares, sticks, and dots on the board to model base-10 blocks. She does not make an explicit connection between the drawings and the blocks. ● The lesson includes a graduated sequence of representations (concrete, representational, abstract).	Examples: ● The teacher is asking students to model two- and three-digit numbers with base-10 blocks. She does not have a clearly sequenced presentation. She gives a number and asks students to model it, draw it, say it all at once. ● The teacher begins a topic with a representation that is too abstract for students. Students are unable to talk about or use

abstraction. Each representation is clearly connected to the previous one. ● The teacher asks students to count by tens. She records the numbers on the board and points out the pattern in the tens place. Then she asks students what is 10 more or 10 less than specific numbers recorded on the board before including numbers with different values in the ones place.	However, these are not presented and connected in a systematic way. When students are confused, she breaks the topic down into an step-by-step progression. ● The teacher asks students to count by tens. She records the numbers on the board and points out the pattern of the digits in the tens place. However, then she orally asks students to add 10 or subtract 10 to randomly selected quantities without a systematic or organized sequence of questions.	the representation meaningfully. ● The teacher orally asks students to add 10 or subtract 10 to randomly selected quantities without a systematic or organized sequence of questions or connection to prior knowledge about place value.

Item 6-The teacher engages students in making connections that develop conceptual understanding. For example, connections can be made between: **· concepts and students' background knowledge,** **· contexts and representations,** **· informal and formal representations.** This item ensures that connections are an explicit part of the lesson.These connections help students understand the relationships that make up the concept(s). To be most effective in developing understanding, students need to see the connections, describe them, use them, and apply them in new problems and situations. Engaging students in making these connections also gives students additional opportunities to practice. Engaging students in making connections can take many forms: think-pair-share, practice that involves modeling and/or explaining, asking for verbal or written responses or for gestures or actions, etc.		
Implemented-3	**Partially Implemented-2**	**Not Implemented-1**
The teacher engages students in making connections that develop conceptual understanding. For example, connections can be made between: · concepts and students' background knowledge, · contexts and representations, · informal and formal representations.	The teacher demonstrates connections that develop conceptual understanding but does not engage students, OR the connections are limited due to missed opportunities.	The teacher does not make connections that develop conceptual understanding, OR the connections are confusing or unclear.
Examples: ● The teacher draws ¼ using a circle model. The teacher draws a fraction bar and partitions it into fourths, shading one section. Then the teacher leads a discussion with purposeful questions making the similarities and differences between the two models explicit. ● To teach a new representation for fractions, the teacher uses a context of	Examples: ● The teacher draws ¼ using a circle model. The teacher draws a fraction bar and partitions it into fourths, shading one section. Then the teacher explains why the two models show the same fraction without engaging students in questions or discussion. ● To teach a new representation for fractions, the teacher connects to a previously used	Examples: ●The teacher draws ¼ using a circle model. The teacher draws a fraction bar and partitions it into fourths, shading one section. The teacher states that the representations are showing the same amount but does not explain why. The connection between them is unclear. ● To teach a new representation for fractions,

submarine sandwiches and draws a sandwich and a fraction bar the same length on the board. As they model different fractions, the teacher asks students to describe what was done. She frequently discusses the bar as if it were a sandwich and asks students to do the same.	representation but misses an obvious opportunity to connect the new representation to a context that is used.	the teacher draws a fraction bar on the board. The teacher does not make connections between the representation, students' background knowledge, or a context.

<table>
<tr><td colspan="3">Item 7-The teacher presents a range of examples that is responsive to the needs of the students.
This item examines whether the teacher provides deliberate and sequenced examples to support students ability to understand and eventually generalize the concept appropriately. There is a clear and deliberate progression to the instruction that is responsive to the needs of students. The teacher increases the complexity after ensuring that students are successful at the current level of complexity.</td></tr>
<tr><td>Implemented-3</td><td>Partially Implemented-2</td><td>Not Implemented-1</td></tr>
<tr><td>The teacher presents a range of examples that is responsive to the needs of the students.</td><td>The teacher presents a range of examples that is somewhat responsive to the needs of the students.</td><td>The teacher does not present a range of examples that is responsive to the needs of the students.</td></tr>
<tr><td>Examples:
● The teacher draws ¼ using a circle model. The teacher asks students how many parts in the whole and how many shaded. The lesson continues with the teacher drawing ¼ with a bar model and then a number line. The teacher then includes other simple fractional amounts (e.g., halves, fourths, eighths.)

● For representing numbers with base-10 blocks, the teacher has a set of examples that increase in magnitude. However, the teacher adds additional examples of the same complexity before moving to the next level as appropriate.</td><td>Examples:
● The teacher draws ¼ using a circle model. The teacher asks students how many parts in the whole and how many shaded. The lesson continues in this way with many fractional amounts but using only circle models. The teacher is missing the opportunity to use additional models that will expand students' understanding of fractional parts.

● For representing numbers with base-10 blocks, the teacher has a set of examples that increase in magnitude. The teacher moves larger magnitudes when some students are successful but some students needed support. The teacher could add an additional example before moving to greater magnitudes or differentiate.</td><td>Examples:
● The teacher draws ¼ using a circle model. The teacher asks students how many parts in the whole and how many shaded. The lesson continues in this way with only fourths and halves. Students do not see different representations or other simple fractional amounts (e.g, eighths or thirds) though they demonstrate readiness.

● The teacher has a set of examples that increase in complexity. The students struggle with the first example, but the teacher moves on through each of the progressively more complex problems anyway.</td></tr>
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Item 8-The teacher consistently discusses mathematical ideas with language that is clear, accurate, and precise. This item focuses on the language used by the teacher during the instruction. The teacher uses unambiguous wording and academic terminology based on the students' receptive vocabulary. Students need to have concepts presented with language that is academic, consistent, and appropriate for the students if they are to use and apply those concepts.		
Implemented-3	**Partially Implemented-2**	**Not Implemented-1**
The teacher consistently discusses mathematical ideas with language that is clear, accurate, and precise.	The teacher discusses mathematical ideas with language that is clear, accurate, and precise but not consistently.	The teacher does not discuss mathematical ideas with language that is clear, accurate, and precise.
Examples: ● The teacher uses the phrase "three-fourths" rather than "three out of four" (such as on a linear representation as compared to discrete items). The teacher is consistent in using terms such as parts, unit size, and whole. ● In response to a student, the teacher says, "Yes, we are doing 'timesing'. In math, the name for 'timesing' is 'multiplying'." The teacher continues to use the word "multiplying" and reminds student of the term when they use "timesing." ● When discussing angles, the teacher uses gestures and says, "The arc drawn near the vertex of the angle is a symbol we use to indicate the size of the angle. Larger angles have longer arcs because the curve travels across a wider opening."	Examples: ● The teacher is consistent with some fraction terms (parts, unit size, whole) but not with others. The teacher says "three out of four" when the context is "three-fourths" (such as on a linear representation). ● In response to a student, the teacher says, "Yes, we are doing 'timesing'. That is the same as multiplying." The teacher does not consistently use the term 'multiplying' later in the lesson. ● When discussing angles, the teacher says, "You see the arcs in there. The larger the arc, the bigger the angle." (imprecise)	Examples: ● The teacher discusses fractions with inconsistent language, switching between terms without clarifying meaning: parts, pieces, whole, one, size of whole, etc. ● In response to a student, the teacher says, "Yes. We are 'timesing'." The teacher uses the word 'timesing' throughout the lesson. ● The teacher uses the context of pies to discuss pi. This language appears to have created confusion for the students. ● When discussing angles, the teacher says, "You see little circles in there. The larger the circle, the bigger the angle." (inaccurate and imprecise)

Item 9- The teacher clearly and sufficiently verbalizes and models reasoning (i.e., think-aloud). This item focuses on how the teacher makes the concepts and the reasoning behind them explicit and clear for the students. This is accomplished by using visual representations, modeling actions, and by carefully and thoroughly articulating the thinking processes involved in the example. Modeling may include gestures as appropriate (sweep of hand to different sides of an equation). This item does not include engaging in questioning of the students; this item is focused on the teacher clearly and sufficiently communicating thought processes.		
Implemented-3	**Partially Implemented-2**	**Not Implemented-1**
The teacher clearly and sufficiently verbalizes and models reasoning (i.e., think-aloud).	The teacher verbalizes and models reasoning but not clearly and/or sufficiently.	The teacher does not verbalize and model reasoning, OR the teacher's reasoning is confusing or inaccurate.
Examples: ● Using base-10 blocks, the teacher counts the ten rods individually and counts by tens. She explains how she knows these 10 rods can be counted both ways. The teacher then shows how these rods fit perfectly on a hundreds block and explains that is how she knows they can trade 10 rods for 1 hundred block. ● The teacher explains that 2/3 and 4/6 are equivalent because they are measures of the same quantity using different size units. The teacher explains how she knows this by lining up two bar models, describing how she partitions the bars, colors in the number of units needed, and sees that equal portions are shaded.	Examples: ● The teacher demonstrates trading base-10 blocks to show quantities two different ways. The explanation is delivered quickly. It is not clear when the words "ten" and "one" refer to the particular size block or to a quantity. ●The teacher explains that 2/3 and 4/6 are equivalent because they are the same size and shows how to draw the fractions using two circle models. The teacher does not sufficiently explain the thinking that she engages in to verify they are the same.	Examples: ● The teacher explains that she can trade a ten rod so she can have more ones. She does not ever state why this works. ● The teacher states, "2/3 and 4/6 are equivalent because they are the same size." There is no explanation or model to support the statement.

Item 10-The teacher provides students with sufficient opportunity to verbalize their understanding and/or reasoning. This item assesses whether students are given an opportunity to communicate their understanding and reasoning. This goes beyond simply providing an answer to a math problem. While teachers may scaffold this by modeling explanations and analyzing their own and others thinking, this item looks specifically at the opportunity for students to communicate their own thinking. This might be done with the aid of visual representations. Opportunities to verbalize may include asking students to think-aloud, summarize, answer questions, agree/disagree, explain or elaborate. Students may verbalize aloud or in writing.		
Implemented-3	**Partially Implemented-2**	**Not Implemented-1**
The teacher provides students with sufficient opportunity to verbalize their understanding and/or reasoning.	The teacher provides students with limited opportunity to verbalize their understanding and/or reasoning.	The teacher does not ask students to verbalize their understanding and/or reasoning.
Examples: ● The teacher frequently asks students to explain their reasoning rather than just provide an answer. For example, she says, “Tell me your thought process here,” and when a student gives a one-word answer, she presses by asking “why?” ● The teacher asks questions and checks for understanding. The teacher then provides another example and asks students to explain their thinking process to one another after solving. The teacher prompts students to explain their thinking.	Examples: ● The teacher asks students to tell her what they did with the manipulatives but does not ask them to explain their reasoning. ● There is some questioning about reasoning during the lesson that enables students to respond orally. However, the teacher provides most of the discussion and examples. ● The teacher asks questions and checks for understanding of the students who offer an answer but not for other students.	Examples: ●The teacher asks for thumbs up/thumbs down. This is done throughout the lesson, but this is the only means students have for communicating their understanding. ● The teacher asks students for answers, but does not ever ask, “How do you know?” ● The teacher calls on students, but feeds them the answers to a degree that it isn’t clear how much students are able to answer on their own.

Item 11-The teacher encourages students to use mathematical vocabulary and/or symbols throughout the lesson. This item assesses whether the teacher provides opportunities for students to be actively engaged with the terminology and symbols that are important to the subject of mathematics. Opportunities to respond should occur frequently throughout the lesson and may be aloud or in writing.		
Implemented-3	**Partially Implemented-2**	**Not Implemented-1**
The teacher encourages students to use mathematical vocabulary and/or symbols throughout the lesson.	The teacher encourages students to use mathematical vocabulary and/or symbols but not consistently throughout the lesson.	The teacher does not encourage students to use mathematical vocabulary and/or symbols.
Examples: ● The teacher defines the term “area” and asks students to repeat the term. Throughout the lesson, the teacher encourages students to use the term “area” accurately and appropriately. ● The teacher consistently prompts students to answer “10 ones” instead of just saying “ten.”	Examples: ● The teacher defines a term “area” and asks students to repeat the term. Though the teacher compliments students who use the term throughout the lesson, she does not consistently ask students to use the term “area” when appropriate. ● The teacher sometimes accepts an answer of “10” instead of “10 ones.”	Examples: ● The teacher defines a term “area” but never asks students to use it. ● The teacher accepts an answer of “10” instead of “10 ones” regularly.

<table>
<tr><td colspan="3">Item 12-The teacher provides students with practice adequate to supporting the development of conceptual understanding.
This item assesses whether students have the opportunity to practice the concept(s) that are the focus of the lesson. To support conceptual development, students should practice with a range of examples that is appropriate for their skill level. A range of examples can include may include number sets with increasing difficulty, as appropriate. The examples may also include translating between representations. Also, the tasks that are involved in practice should reinforce connections and employ the visual representations that were included in the lesson.</td></tr>
<tr><td>Implemented-3</td><td>Partially Implemented-2</td><td>Not Implemented-1</td></tr>
<tr><td>The teacher provides students with practice adequate to supporting the development of conceptual understanding.</td><td>The teacher provides students with practice somewhat adequate to supporting the development of conceptual understanding.</td><td>The teacher provides students with practice inadequate to supporting development of conceptual understanding.</td></tr>
<tr><td>Examples:
● Practice asks students to state the multiplication sentence given an array and also asks students to make an array given the multiplication sentence.

● After teaching students how to use an area model to multiply 2 digit numbers, students practice with a worksheet that provides space and prompts for constructing an area models.

● Students begin guided practice but are struggling due to complexity of the task. The teacher adjusts and provides simpler practice focused on the concept at which students are successful.</td><td>Examples:
● Practice only involves stating the multiplication sentence given an array and does not include making arrays given the multiplication sentence.

● After teaching students how to use an area model to multiply 2 digit numbers, students practice with a worksheet that only includes limited opportunity to apply the strategy.

● Students begin guided practice but are struggling due to complexity of the task. After growing confusion, the teacher adjusts and provides scaffolds that support their ability to complete the task as given. The scaffolding emphasizes a procedure for completing the task, and therefore the conceptual goal becomes implicit.</td><td>Examples:
● Practice shifts to procedure of solving simple multiplication problems before conceptual understanding of multiplication as an array has been established.

● The teacher allows students to choose the numbers they will model. Many students choose number too complex for them to model successfully. The teacher does not adjust.

● Rather than providing scaffolds when students are having difficulty, the teacher tells the students what numbers go in the blanks on their worksheets.</td></tr>
</table>

Item 13-The teacher uses questions and prompts related to visual representations to encourage students to monitor and check their work. This item examines questions and prompts used by the teacher as guidance. This guidance supports students' ability to use and reason with the visual representation on future tasks. Guidance is an interaction between the teacher and students. The teacher provides students with adequate support as they reason about a visual representation. The types of questions or prompts needed may vary depending upon the complexity and nature of the question or task and the students' current level of understanding and should be adjusted accordingly.		
Implemented-3	**Partially Implemented-2**	**Not Implemented-1**
The teacher uses questions and prompts related to visual representations to encourage students to monitor and correct their work.	The teacher uses questions and prompts related to visual representations but not effectively.	The teacher does not use questions and prompts related to visual representations.
Examples: ● The teacher asks a student to look at the denominators of the equivalent fractions modeled on the fraction bars. She questions and prompts the student to use the fraction bars to find something equivalent to 7/12. ● The teacher provides coaching on using the number line for multiplication, and consistently reminds students to identify the groups and size of groups.	Examples: ● The teacher asks a student to look at the denominators of the equivalent fractions modeled on the fraction bars. She walks away without guiding the student in how to use this observation to find something equivalent to 7/12. ● The teacher reminds students to use a number line but does not monitor their use to provide support as needed.	Examples: ● The teacher does not prompt the student to use the fraction bars on the desk when looking for something equivalent to 7/12. The teacher asks the student to think about the factors of 12. ● The teacher uses a number line to present rounding but does not prompt students to use a number line as they practice.

Item 14-Feedback is consistently linked to mathematical reasoning and concepts. This item evaluates the focus and the wording feedback. Feedback that is specific and informative provides students with information directly linked to their progress. It may include specific information about reasoning, processes, or calculations. Feedback can take the form of correction, suggestion, prompting, cueing or reinforcing and affirming. Both the nature and complexity of the task and the lesson goal need to be taken into consideration. Affirmative or corrective feedback that is immediate can reinforce accuracy, prevent misconceptions, or ensure accurate practice. Somewhat delayed feedback that allows the students time to think through the steps of a complex process may also be appropriate and support confidence, independence and self-regulation. Opportunities for specific feedback on reasoning or concepts may occur with practice of new tasks, after an error, or reinforcing correct thinking. When misconceptions arise, feedback should identify the incorrect reasoning, clarify the correct reasoning, and reinforce students' use of the correct reasoning.		
Implemented-3	**Partially Implemented-2**	**Not Implemented-1**
Feedback is consistently linked to mathematical reasoning and concepts.	Feedback is not consistently linked to mathematical reasoning and concepts.	There is no feedback, OR feedback is not linked to mathematical reasoning and concepts.
Examples: ● When the student sets up the counters to correctly show 2x3, the teacher reinforces by counting the groups and connecting back to the context of the problem. When the student misses a group in 2 x 6, the teacher provides prompts that support correctly setting up the problem. ● Students are successful with the tasks. The teacher takes several opportunities to give informative feedback such as, "Good. You saw that there were five parts in a whole, so you took out groups of five."	Examples: ● Feedback is specific and informative when students make a mistake but affirmative feedback is not specific. For example, a student models 6 + 2 instead of 6 x 2. The teacher gives corrective feedback. When the student is successful with the next problem of 4 x 3, the teacher' just says, "You got it." The teacher could say, "Good. You knew that you were modeling groups and size of group."	Examples: ● The teacher consistently only tells students if they are right or wrong. If they are wrong, the teacher says, "Read it again." ● The teacher simply states that students are correct or prompts students to perform actions to model simple multiplication. The teacher does not discuss the meaning of the actions. ● Students are successful with the tasks. The teacher only tells students, "Good job."

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