Understanding Procedures in Mathematics Rubric Manual

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Understanding Procedures in Mathematics Rubric

Overview:

Understanding of procedures in mathematics refers to knowing the conceptual basis for the procedure and the reasons for the steps in the procedures. The conceptual basis for procedures and the ability to accurately apply procedures are both important; both support the ability to select and use an appropriate procedure to solve mathematical problems (Baroody, Feil, & Johnson, 2007; Braithwaite, Tian, & Siegler, 2017; Rittle-Johnson, Schneider, & Star, 2015). When students are taught in a manner that includes understanding of procedural skills, they show greater achievement and retention than those who receive instruction focused on the steps of a procedures alone (Geary, et al., 2008; Rittle-Johnson, Schneider, & Star, 2015). These outcomes have occurred with mathematics topics from early computation to algebra (e.g., Butler, Miller, Crehan, Babbitt, & Pierce, 2003; Flores, 2010; Fuchs et al., 2016; Shanley, Clarke, Doabler, Kurtz-Nelson, & Fien, 2017; Witzel, Mercer, & Miller, 2003). Understanding of procedures is more efficient than rote memorization because it applies across many novel situations. It increases the likelihood that students will apply what they've learned in new situations, both in and out of school (Rittle-Johnson & Alibali, 1999).

The Understanding Procedures Rubric was designed for use by supervisors and administrators to reliably evaluate teachers' implementation of practices that support the development of students' understanding of mathematics procedures. The rubric provides specific, accurate, and actionable feedback to special education teachers about the quality of their instruction, and ultimately, improve the outcomes for students with disabilities. The purpose of this manual is to provide technical information for implementing the Understanding Procedures Rubric rubric as a tool for evaluation and feedback.

This rubric includes 5 components. These are:
1) Content of Instruction
2) Design of Instruction
3) Delivery of Instruction
4) Student Engagement
5) Providing Feedback

Under these 5 components, there are 17 items. For each item, there are five levels of implementation. Observing either live or from video, the observer assigns a rating based on a scale that ranges from **Implemented** to **Not Implemented.** The rater selects one score from among the following choices:
3--Implemented, meaning the teacher's performance aligns with the descriptor,
2+
2--Partially Implemented, meaning the teacher's instruction reflects this item but there are flaws or missing components in the way in which it is implemented,
2-
1--Not Implemented, meaning the item is either implemented poorly or should have been observed but is not,
and **Not Applicable (N/A)**, a category that recognizes that given the lesson context and what is taught in previous lessons, not every item will be observed across every observation.

Preparation for the Observation

There are several materials you will need in order to use the Understanding Procedures Rubric to conduct the observations. First, you should ensure you have everything you need to conduct the observation including pencils, a clipboard (or something hard to write on), and a copy of the Understanding Procedures Rubric. The Understanding Procedures Rubric is your scoring form and your note-taking space. Use the margins and the backs of pages to write notes of the things you observe that help you determine what ratings to assign. The notes will be also useful when you need to provide feedback to the teacher.

Understanding the Understanding Procedures Rubric Structure

There are 17 items in this rubric. Each item is listed in a table below with an explanation and description of the intention of the item to help clarify its meaning. Each item has five levels of implementation. Descriptors are given for high, middle, and low levels of implementation. Examples are included to help you interpret the meaning of the different implementation levels. You should consider these descriptions and examples as you determine the implementation level for each item.

Assigning Rating on Understanding Procedures Rubric

The Understanding Procedures Rubric rating scale includes a score of: **1)"Not Implemented," 2) "Partially Implemented," and 3) "Implemented,"**. The "Partially Implemented" category is further divided to allow for assigning a **2-, a 2, or a 2+**, to indicate the degree to which the item is partially implemented. A **2-** indicates a very low level of partial implementation, whereas a **2+** can be used in cases where the item is almost fully implemented but not quite.

Observing either live or from video, you assign a rating on the basis of the observations. Assign a rating that comes closest to describing the observation even if not an exact match. For each item, assign a single rating, unless it is N/A.

Because the duration of a class may be 40 minutes or more, it is helpful to note whatever is observed, even at a low level. Then if a higher level item implementation makes the previous item inaccurate, the previous choice can simply be changed. This is especially useful when some items need to be observed throughout the whole lesson. For example, after observing the teacher "allows adequate time for students to think or respond," the observer should check 'Partially Implemented', but if the teacher continues to allow adequate time for students to think or respond in that way until the end of the lesson, 'Partially Implemented' should be crossed out and a higher level of item implementation is checked.

Description of the Understanding Procedures Rubric rubric

Component 1: Content of Instruction

The purpose of this section is to describe aspects of mathematics lesson content that research has identified as effective in Understanding Procedures in mathematics.

Elements of Component 1 are:

Item 1- The lesson is consistently focused on understanding of critical mathematics procedures (amount appropriate for meaningful development).

Item 2-The teacher uses visual representations that support understanding of the procedure, e.g., representations show the appropriate size relations, map to meaning of numbers when in context, and map directly to algorithm.

Component 2: Design of Instruction

This component contains items that describe the way a lesson is structured. Rather than focusing on the content of the lesson, this is focused on how a teacher has planned to systematically present the material while making appropriate connections to previously learned material, elements within the lesson, and students' knowledge.

Elements of Component 2 are:

Item 3- The teacher reviews and checks that students have the conceptual background knowledge and skills necessary for understanding the procedure.

Item 4-To begin instruction, the teacher selects a simple or familiar context or representation that effectively provides meaning for the procedure and/or operation.

Item 5-The teacher effectively reviews or teaches key vocabulary and/or symbols.

Item 6-There is an explicit systematic progression within and/or across lessons, e.g., a step-by-step presentation, a graduated sequence of representations, or explicit connections to previous lessons.

Item 7-The teacher clearly explains and sufficiently emphasizes the conceptual meaning of the procedure and/or operation.

Item 8-The teacher engages students in making connections between representations, meanings of operations, and procedures.

Component 3: Delivery of Instruction

This component contains items that describe the manner in which the teacher delivers the instruction. This includes the ways in which the teacher is responsive to students' needs and the quality of the teacher's communication.

Elements of Component 3 are:

Item 9-The teacher provides clear explanations for all of the mathematical reasons for the steps in the procedure.

Item 10-The teacher presents a range of examples that is responsive to the needs of the students.

Item 11-The teacher consistently discusses mathematical ideas with language that is clear, accurate, and precise.

Item 12-The teacher clearly and sufficiently verbalizes and models reasoning (i.e., think-aloud).

Component 4: Student Engagement

This component contains items that describe how the teacher has planned for and implements opportunities for students to be engaged in the lesson and practice working with the concept.

Elements of Component 4 are:

Item 13-The teacher provides students with sufficient opportunity to verbalize their understanding and/or explain their reasoning.

Item 14-The teacher encourages students to use mathematical vocabulary and/or symbols throughout the lesson.

Item 15-The teacher provides students with practice adequate to supporting the development of understanding of the procedure.

Component 5: Providing Feedback

This component contains items that describe the nature of the feedback provided to students.

Elements of Component 5 are:

Item 16-The teacher uses questions and prompts related to visual representations, strategies, rules, and/or application of memory techniques to encourage students to develop independence.

Item 17-Feedback is consistently linked to mathematical reasoning and concepts.

Psychometric Properties

The Understanding Procedures Rubric rubric has been developed through a rigorous process to ensure that it is a valid and reliable instrument. Each item included within the rubric comes from an analysis of the existing research establishing these instructional practices as effective for students with high incidence disabilities. Additionally, the rubric has been reviewed by content experts in the field to support content validity.

Further psychometric review is ongoing.

Key Terms on the Understanding Procedures Rubric

“**Adequate**” means as much or as good or as necessary to accomplish a purpose or produce intended or expected results.
“**Consistently**” means every time the opportunity arises, the teacher responds in the same or an appropriately similar way. It is different from continuously.
“**Effectively**” means adequate to accomplish a purpose or produce intended or expected results.
“**Frequently**” means regularly or often.
“Sufficiently” means enough to accomplish a purpose.
“Visual representations” refer to concrete and pictorial representations of numbers, concepts, and relationships. Pictorial representations can be informal or formal. For the purposes of this rubric, visual representations include, but are not limited to, concrete manipulatives, concrete or pictorial bar models, tape diagrams, pictures, icons, number lines, graphs, etc.

Item Descriptions and Examples

Item 1: The lesson is consistently focused on understanding of critical mathematics procedures (amount appropriate for meaningful development).
This item addresses the need for students to understand the foundational procedures that will support their success with the general education curriculum in current and future years. Critical math procedures are those that help student reach the most critical standards or big ideas. In early grades, these critical procedures involve counting, operations, and working with the base-10 place value system Understanding these procedures also involves knowing the reasoning underlying the algorithms. In later grades, these procedures extend to solving problems with fractions, decimals, ratios, and percents, and, again, knowing the reasoning underlying the algorithms. Once these foundational procedures are developed, the focus turns to integers and algebra. A lesson may include one or more procedures. If more than one procedure is addressed or practiced, there should not be so many that the students are confused or struggling to understand them all.

Implemented-3	Partially Implemented-2	Not Implemented-1
The lesson is **consistently** focused on understanding of critical mathematics procedures (**amount appropriate** for meaningful development).	The lesson addresses understanding of critical procedures but **not consistently**, OR the lesson addresses **too many** procedures for meaningful development.	The lesson is **not** focused on understanding procedures, OR the lesson does **not** address critical content.
Examples: ● The lesson maintains focus on the meaning of multiplying by 10 and by 100. All modeling, think-aloud, and tasks are related to the powers of 10, and the teacher maintains focus on the relationship between adding zeros and the value of the number. ● The teacher maintains a	Examples: ● The lesson begins with the meaning of multiplying by 10 and by 100. All modeling, think-aloud, and initial examples are related to the powers of 10. However, as it progresses, the lesson and tasks turn to adding zeros to the end of the number and no longer focuses on why. ● The lesson includes instruction on both addition to 20 and place	Examples: ● The lesson is focused on multiplying by 10 and by 100. All presentation is focused on learning the rule of adding zeros to the number without discussion of the meaning of doing so. ● The lesson does not move beyond an activity in which students build

consistent focus on addition to 20 by selecting a task and context that emphasize the concept of addition. There are no digressions. ● The teacher uses students' prior knowledge of perimeter to provide context for a lesson on the procedure for adding fractions. The context enables consistent focus on meaning for adding fractions.	value and names of teen numbers. There are too many important ideas at once for students to develop a strong understanding of them. ● The teacher allows a student comment to lead to a long discussion of another unrelated mathematics topic rather than integrating students' knowledge into the topic or stopping the discussion early.	shapes with blocks. ● The lesson is focused on telling time with an analog clock.

Item 2-The teacher uses visual representations that support understanding of the procedure, e.g., representations show the appropriate size relations, map to meaning of numbers when in context, and map directly to algorithm. The purpose of this item is to ensure that visual representations are used in the lesson. The representations also need to be consistent with the concepts that are the basis for the procedure. Visual representations should meaningfully represent the quantities that are linked to the numbers. Visual representations can include manipulatives, simple drawings, number lines, bar models, tape diagrams, arrays, graphs etc.		
Implemented-3	**Partially Implemented-2**	**Not Implemented-1**
The teacher uses visual representations that support understanding of the procedure, e.g., representations show the appropriate size relations, map to meaning of numbers when in context, and map directly to algorithm.	The teacher uses visual representations that **somewhat** support understanding of the procedure, OR **some** of the visual representations support understanding of the procedure.	The teacher does **not** use visual representations, OR the teacher uses visual representations that do **not** support understanding of the procedure.
Examples: ● The teacher uses base-10 blocks and drawings of them that reflect the 10-to-1 relationship. ● During a lesson on adding fractions, the teacher uses a number line to represent a context of running a relay race. ● The teacher uses open arrays to model partial-products multiplication. The arrays are drawn to reflect the sizes of the decomposed numbers.	Examples: ● The teacher uses base-10 blocks, but the drawings of these blocks on the board do not reflect the 10-to-1 relationship. ● During a lesson on adding fractions, the teacher uses a combination of circle models and a number line to represent a context of running a relay race. ● The lesson involves constructing open arrays to model partial-products multiplication. The arrays are not drawn to reflect the sizes of the decomposed numbers.	Example: ●The teacher uses objects to represent hundreds, tens, and ones that are not proportional (do not reflect the 10 to 1 relationship). ● During a lesson on adding fractions, the teacher uses circle models to represent a context of running a relay race. ● The teacher presents a lesson on partial-products multiplication relying only on writing numbers on the board. There are no visual models.

Item 3-The teacher reviews and checks that students have the conceptual background knowledge and skills necessary for understanding the procedure. This item assesses whether the teacher verifies that students have the necessary background knowledge and prior skills that support development of understanding of the procedure. This is not the same as review for practice of previously taught skills. This review should be focused on the understanding and skills needed to be successful in the coming lesson rather than on a broadly related topic.		
Implemented-3	**Partially Implemented-2**	**Not Implemented-1**
The teacher reviews **and** checks that students have the conceptual background knowledge and skills necessary for understanding the procedure.	The teacher reviews **but** does not check that students have the conceptual background knowledge and skills necessary for understanding the procedure.	The teacher does **not** review **or** check that students have the conceptual background knowledge and skills necessary for understanding the procedure.
Examples: ●Prior to beginning a lesson on adding fractions with like denominators on a number line, the teacher reviews by locating two fractions on a number line. She also asks students to demonstrate locating an additional fraction on the number line. ● The teacher has students solve a two-digit multiplication problem with an open array. She verifies students are successful before connecting the arrays to demonstrating partial-products multiplication with numbers only.	Examples: ●The teacher begins the lesson a review of placing two fractions on a number line. The teacher does not check that students are able to locate fractions on the number line. ● The teacher solves a two-digit problem with an open array on the board. She asks students some guiding questions but doesn’t verify they remember or fully understand the representation or process before connecting to multiplication with numbers only.	Examples: ●.The teacher reviews locating fractions on a number line after instruction has begun and students are demonstrating difficulty with the concept. ●The teacher asks students to solve a single digit multiplication problem. The teacher does not check that students understand multiplication with an open array before beginning to connect arrays to partial-products multiplication with numbers.

Item 4-To begin instruction, the teacher selects a simple or familiar context or representation that effectively provides meaning for the procedure and/or operation. This item examines how effectively the context or representation connects to what students already know. The teacher provides an initial context for the numbers or initially uses a visual representation (manipulative or model) that is simple or familiar enough for students to engage with meaningfully from the start of the lesson without the context or representation creating confusion. For example, a simple drawing can help students make sense of complex procedures such as operations with multi-digit numbers or fractions.		
Implemented-3	**Partially Implemented-2**	**Not Implemented-1**
To begin instruction, the teacher selects a simple or familiar context or representation that **effectively** provides meaning for the procedure and/or operation.	To begin instruction, the teacher selects a context or representation that does **not effectively** provide meaning for the procedure and/or operation.	The teacher does **not** begin instruction with a context or representation, OR the teacher begins instruction with a context or representation that is **confusing or inaccurate.**
Examples: ● The teacher begins a lesson on division of fractions with bar models that show ¼, ½, and ¾. Students are familiar with this representation. Bar models are effective for showing division. ● To begin a lesson on division of fractions, the teacher begins with a context of dividing 2 ½ bags of potting soil into various numbers of flower pots. (simple context that provides meaning for numbers)	Examples: ● The teacher begins a lesson on division of fractions by handing out plastic fraction bars that are not divisible. Students have to locate bars with the correct number of partitions from among the set to compare to the original. ● To begin a lesson on division of fractions, the teacher creates a story about making lunches for a field trip that involves dividing large boxes of differing quantities of several foods. (unnecessary complexity reduces effectiveness)	Examples: ● The teacher begins a lesson on division of fractions. The teacher does not use a context or representation. ● The teacher begins a lesson on division of fractions with a context that really is multiplication of fractions. ● The teacher begins the lesson with a context unfamiliar to the students and creates confusion.

Item 5-The teacher effectively reviews or teaches key vocabulary and/or symbols. It is important to ensure that the meanings of vocabulary and symbols are clear to students. Ensuring this clarity is important for students with disabilities such as memory or language processing rather than assuming they will remember or infer the meanings. This may occur at the beginning of a lesson or may occur mid-way through a lesson as appropriate. Students may provide the review if they are able to provide clear and accurate definitions and/or examples. If they cannot, the teacher should provide clear and explicit definitions. Effectively reviewing or teaching involves a clear, timely, concise, focused explanation of the term or symbol.		
Implemented-3	**Partially Implemented-2**	**Not Implemented-1**
The teacher **effectively** reviews or teaches **key** vocabulary and/or symbols.	The teacher reviews or teaches key vocabulary and/or symbols but **not effectively**, OR the teacher reviews or teaches **some** key vocabulary and/or symbols.	The teacher does **not** review or teach key vocabulary and/or symbols.
Examples: ● After stating they will be adding fractions with unlike denominators, the teacher quickly verifies that students remember key vocabulary (numerator, denominator, and equivalence) by asking for a definition and providing a concise definition when students do not remember. ● After an activity in which students use cubes to model a simple context involving exponents, the teacher clearly explains the terms (base and exponent), linking the definitions to the activity, and clearly models the symbolic notation.	Examples: ● After stating they will be adding fractions with unlike denominators, the teacher spends several minutes reviewing vocabulary words by providing hints and having student try to guess the meaning. ● During a lesson on adding with unlike denominators, the teacher reviews the terms numerator and denominator after discovering students are confused about the terms. ● After an activity in which students use cubes to model a simple context involving exponents, the teacher explains the terms (base and exponent) and the symbolic notation without reference to the activity.	Examples: ● The teacher presents a lesson on adding fractions with unlike denominators without a review of important vocabulary such as numerator, denominator, and equivalence. ● The teacher leads students through an activity modeling exponents but does not provide any definitions or explanations of terms or symbolic notation.

Item 6-There is an explicit systematic progression within and/or across lessons that supports understanding, e.g., a step-by-step presentation, a graduated sequence of representations, or explicit connections to previous lessons. This item assesses whether the teacher organizes instruction of the procedure into a logical sequence to help students build connections that support understanding. Systematic instruction can be a sequential presentation of steps, organized presentation of examples and non-examples, or an organized compare and contrast. Systematic instruction may teach students to develop representations and gradually formalize these, transitioning to abstract symbols over time. Regardless of the structure, it is logical, clear, organized, and provides unambiguous connections.		
Implemented-3	**Partially Implemented-2**	**Not Implemented-1**
There is an explicit systematic progression within and/or across lessons that supports understanding, e.g., a step-by-step presentation, a graduated sequence of representations, or explicit connections to previous lessons.	There is a **somewhat** explicit or somewhat systematic progression within and/or across lessons, OR there is an explicit, systematic progression within and/or across lessons that **somewhat** supports understanding.	There is **not** an explicit or systematic progression within or across lessons that supports understanding.
Examples: ● The teacher begins the lesson by connecting to the previous day's lesson using base-10 blocks to model subtraction of two-digit numbers. In this lesson, students model subtraction of three digit numbers with base-10 blocks. ● The lesson on the multiplication algorithm is clearly connected to a graduated sequence of representations (concrete, representational, abstract) that are have been	Examples: ● The teacher states they will build on yesterday's activity of subtracting three-digit numbers with base-10 blocks. She begins the lesson by drawing squares, sticks, and dots on the board. She does not make an explicit connection between the drawings and the blocks. ● The teacher presents a lesson on the multiplication algorithm with several demonstrations of the whole procedure. Initially, these are not presented in a clear	Examples: ● The teacher is asking students to model subtraction of two-digit numbers with base-10 blocks. She does not have a clearly sequenced presentation. She gives a number and asks students to model it, draw it, and write it in numerals all at once. ● The teacher begins a topic with a representation that is too abstract for students. Students are unable to talk about or use the representation meaningfully.

presented in previous lessons. ● The teacher presents a lesson on the multiplication algorithm with a clear presentation of a step-by-step sequence of actions.	step-by-step manner. When students are confused, she breaks the topic down into an explicit step-by-step progression.	● The teacher presents a lesson on the multiplication algorithm with several demonstrations of the whole procedure. These are not presented in a clear step-by-step manner. When students are confused, the teacher presents more examples in the same way-never making the step-by-step process explicit.

Item 7-The teacher clearly explains and sufficiently emphasizes the conceptual meaning of the procedure and/or operation. This item assesses whether the teacher makes the conceptual basis for the procedure, as well as the meaning of any related operation (addition, subtraction, multiplication, or division), a clear and explicit part of the instruction. This supports students' abilities to understand the steps of the procedure and use the procedure appropriately in the future.		
Implemented-3	**Partially Implemented-2**	**Not Implemented-1**
The teacher **clearly** explains and **sufficiently** emphasizes the conceptual meaning of the procedure and/or operation.	The teacher explains the conceptual meaning of the procedure and/or operation but **not clearly** or with **insufficient** emphasis.	The teacher does **not** explain the conceptual meaning of the procedure and/or operation, OR the explanation is **confusing or inaccurate**.
Examples: ●.The teacher maintains a focus on addition to 20 by consistently emphasizing one meaning of addition as joining two groups and finding the size of the new group. ●. The teacher maintains a consistent focus on the meaning of division during a lesson on division with fractions. The teacher carefully and clearly contrasts the meaning of division with the meaning of multiplication.	Examples: ● The teacher presents a lesson on addition to 20 but does not sufficiently emphasize the meaning of addition. There are several missed opportunities to emphasize the meaning of the operation. ● The teacher maintains a focus on the meaning of division during a lesson on division with fractions. However, the explanation of the difference between division and multiplication is not clearly explained.	Examples: ● The teacher presents a lesson on addition to 20 with no discussion of the meaning of addition. The lesson focuses on the process of counting on without any mention of why this is addition. ●The teacher does not use any language related to the meaning of division while modeling division of fractions.

Item 8-The teacher engages students in making connections between representations, meanings of operations, and procedures. This item ensures that connections are an explicit part of the lesson. To be most effective in developing understanding, students need to see the connections, describe them, use them, and apply them in new problems. Engaging students in making these connections also gives students additional opportunities to practice. Engaging students in making connections can take many forms: think-pair-share, practice that involves modeling and/or explaining, asking for verbal or written responses or for gestures or actions, etc.		
Implemented-3	**Partially Implemented-2**	**Not Implemented-1**
The teacher engages students in making connections between representations, meanings of operations, and procedures.	The teacher demonstrates connections between representations, meanings of operations, and procedures **but** does not engage students, OR the connections are **limited** due to missed opportunities.	The teacher does **not** make connections between representations, meanings of operations, and procedures.
Examples: ● The teacher draws an open array and partial-products multiplication on the board. Then the teacher leads a discussion with purposeful questions making the similarities and differences between the two solution methods explicit. ● To teach the procedure for finding equivalent fractions, the teacher clearly models the connections between representations with fraction bars and the algorithm. Then the teacher asks students to show these connections to a partner with another example.	Examples: ● The teacher draws an open array and partial-products multiplication on the board. Then the teacher explains why the two solution methods are similar. ●The teacher makes connections between partial-products and an open array but misses obvious opportunities to connect to the context being used. ● To teach the procedure for finding equivalent fractions, the teacher clearly models the connections between representations with fraction bars and the algorithm. The teacher does not ask students to discuss the representations or identify connections.	Examples: ● The teacher draws an open array and partial-products multiplication on the board. Then the teacher tells students the methods get the same answer with no explanation. ● To teach the procedure for finding equivalent fractions, the teacher draws fraction bars on the board and then demonstrates the procedure. The teacher does not make connections between the representation. The relationship between the model and procedure is left implicit.

<table>
<tr><td colspan="3">Item 9-The teacher provides clear explanations for all of the mathematical reasons for the steps in the procedure.
This item focuses on whether and how well the teacher explains the procedure for the students. The explanations should include the logic or reasoning for each step in the procedure. Also, the explanations should clearly show the students what procedure looks like when it is performed proficiently.</td></tr>
<tr><td>Implemented-3</td><td>Partially Implemented-2</td><td>Not Implemented-1</td></tr>
<tr><td>The teacher provides clear explanations for all of the mathematical reasons for the steps in the procedure.</td><td>The teacher provides explanations for some of the the mathematical reasons for the steps in the procedure,
OR
the explanations are not always clear.</td><td>The teacher does not provide explanations for the mathematical reasons for the steps in the procedure,
OR
the explanations are confusing or inaccurate.</td></tr>
<tr><td>Examples:

● The teacher explains the mathematical reasons for rules for rounding numbers in an organized, student-friendly way with clear examples on a number line.

● The teacher models how to add 2 digit numbers with regrouping clearly and consistently explaining the reason for each step with several examples.</td><td>Examples:

● The teacher provides an overly complicated verbal explanation for mathematical reasons for the rules for rounding numbers.

● The teacher models how to add 2 digit numbers with regrouping clearly but does not provide reasons for all the steps.</td><td>Examples:

● The teacher provides rules for rounding but not the mathematical reasons for the rules.

● The teacher models how to add 2 digit numbers with regrouping clearly but explanations for the reasons for steps seem made up as she goes. The result is a lack of consistency in the explanation to students.</td></tr>
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<table>
<tr><td colspan="3">Item 10-The teacher presents a range of examples that is responsive to the needs of the students.
This item examines whether the teacher provides deliberate, careful, and sequenced examples to support students ability to understand and eventually apply the procedure correctly and appropriately. There is a clear and deliberate progression to the instruction that is responsive to the needs of students. The teacher increases the complexity after ensuring that students are successful at the current level of complexity.</td></tr>
<tr><td>Implemented-3</td><td>Partially Implemented-2</td><td>Not Implemented-1</td></tr>
<tr><td>The teacher presents a range of examples that is responsive to the needs of the students.</td><td>The teacher presents a range of examples that is somewhat responsive to the needs of the students.</td><td>The teacher does not present a range of examples that is responsive to the needs of the students.</td></tr>
<tr><td>Examples:
● The teacher begins a lesson using simple unit fractions including ½, ¼, and 1/8. After checking for understanding, the teacher adds examples of ⅓, ⅕, and ⅙ before moving on to non-unit fractions.

● The teacher has a set of examples of addition with regrouping that increase in complexity (regrouping in ones place with known facts, regrouping in ones place with harder facts, regrouping in tens place). The teacher adds additional examples of the same complexity as needed before moving to the next level.</td><td>Examples:
● The teacher presents a lesson using unit fractions. Though the students demonstrate readiness through success with many examples with larger denominators, the teacher does not add non-unit fractions limiting students' opportunity to develop understanding.

● The teacher has a set of examples that increase in complexity. The teacher gives students adequate practice with regrouping with known facts before moving to harder facts. However, the teacher moves to regrouping in the tens place when a few students needed support with the last example.The teacher could add an additional example before moving to greater complexity or differentiate.</td><td>Examples:
● The teacher has a set of examples that are overly limited in terms of complexity. The task is too easy for students.

● The teacher has a set of examples that increase in complexity. The students struggle with the first example, but the teacher moves on through each of the progressively more complex problems anyway.</td></tr>
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Item 11-The teacher consistently discusses mathematical ideas with language that is clear, accurate, and precise. This item focuses on the language used by the teacher during the instruction. The teacher uses unambiguous wording and academic terminology based on the students' receptive vocabulary. Students need to have concepts presented with language that is academic, consistent, and appropriate for the students if they are to use and apply those concepts.		
Implemented-3	**Partially Implemented-2**	**Not Implemented-1**
The teacher **consistently** discusses mathematical ideas with language that is clear, accurate, and precise.	The teacher discusses mathematical ideas with language that is clear, accurate, and precise but **not consistently**.	The teacher does **not** discuss mathematical ideas with language that is clear, accurate, and precise.
Examples: ● The teacher uses the phrase "three-fourths" rather than "three out of four" (such as on a linear representation as compared to discrete items). The teacher is consistent in using terms such as parts, unit size, and whole. ● In response to a student, the teacher says, "Yes, we are doing 'timesing'. In math, the name for 'timesing' is 'multiplying'." The teacher continues to use the word "multiplying" and reminds student of the term when they use "timesing." ● When discussing angles, the teacher says, "The arc drawn near the vertex of the angle is a symbol we use to indicate the size of the angle. Larger angles have longer arcs because the curve travels across a wider opening."	Examples: ● The teacher is consistent with some fraction terms (parts, unit size, whole) but not with others. The teacher says "three out of four" when the context is "three-fourths" (such as on a linear representation). ● In response to a student, the teacher says, "Yes, we are doing 'timesing'. That is the same as multiplying." The teacher does not consistently use the term 'multiplying' later in the lesson. ● When discussing angles, the teacher says, "You see the arcs in there. The larger the arc, the bigger the angle." (imprecise)	Examples: ● The teacher discusses fractions with inconsistent and imprecise language, switching between terms without clarifying meaning: parts, pieces, whole, one, size of whole, size of parts, etc. ● In response to a student, the teacher says, "Yes. We are 'timesing'." The teacher uses the word 'timesing' throughout the rest of the lesson. ● When discussing angles, the teacher says, "You see little circles in there. The larger the circle, the bigger the angle." (inaccurate and imprecise)

Item 12- The teacher clearly and sufficiently verbalizes and models reasoning (i.e., think-aloud). This item focuses on how the teacher makes the reasoning behind the procedures explicit and clear for the students. This is accomplished by using visual representations, modeling actions, and by carefully and thoroughly articulating the thinking processes involved in the example. Modeling may include gestures as appropriate (sweep of hand to different sides of an equation). This item does not include engaging in questioning of the students; this item is focused on the teacher clearly and sufficiently communicating thought processes.		
Implemented-3	**Partially Implemented-2**	**Not Implemented-1**
The teacher clearly and sufficiently verbalizes and models reasoning (i.e., think-aloud).	The teacher verbalizes and models reasoning but not clearly and/or sufficiently.	The teacher does not verbalizes and models reasoning, OR the teacher's reasoning is confusing or inaccurate.
Examples: ● Using drawings of base-10 blocks to provide a visual, the teacher explains how she knows what to do when she is regrouping. She explains *how she knows* that 1 stick (a ten) is the same as 10 cubes (ten ones). The teacher then shows how the total amount does not change after the trade. She explains her thought processes clearly and for each step. ● The teacher explains how to round numbers with several examples. Each time, the teacher carefully articulates what she looks at, what decisions she makes, and why.	Examples: ● The teacher demonstrates trading base-10 blocks for regrouping. The explanation is delivered quickly. It is not clear when the words "ten" and "one" refer to the particular size block or to a quantity. ●The teacher explains how to round numbers with several examples. Her "think-aloud" is limited to why she rounds up or down. She does not thoroughly explain what she thinks as she first looks at the number or how she knows she is correct.	Examples: ● The teacher explains that she needs to trade a ten rod so she can have more ones. She does not ever state why this works. ●The teacher shows how to round numbers without a "think-aloud" that makes the reasoning one uses transparent.

<table>
<tr><td colspan="3">Item 13-The teacher provides students with sufficient opportunity to verbalize their understanding and/or explain their reasoning.
This item assesses whether students are given an opportunity to communicate their understanding and reasoning. This supports students’ abilities to learn and remember the mathematics, as well as providing the teacher with a check for understanding. Understanding and reasoning goes beyond simply providing an answer to a math problem. While teachers may scaffold this by modeling explanations and analyzing their own and others thinking, this item looks specifically at the opportunity for students to communicate their own thinking. This might be done with the aid of visual representations. Opportunities to verbalize may include asking students to think-aloud, summarize, answer questions, agree/disagree, explain or elaborate. Verbalizations may be aloud or in writing.</td></tr>
<tr><td>Implemented-3</td><td>Partially Implemented-2</td><td>Not Implemented-1</td></tr>
<tr><td>The teacher provides students with sufficient opportunity to verbalize their understanding and/or explain their reasoning.</td><td>The teacher provides students with limited opportunity to verbalize their understanding and/or explain their reasoning.</td><td>The teacher does not ask students to verbalize their understanding and/or explain their reasoning.</td></tr>
<tr><td>Examples:
● The teacher frequently asks students to explain their reasoning rather than just provide an answer. For example, she says, “Tell me your thought process here,” and when a student gives a one-word answer, she presses by asking, “Why?”

● The teacher asks questions and checks for understanding. The teacher then provides another example and asks students to explain their thinking process to one another after solving. The teacher prompts students to explain their thinking.</td><td>Examples:
● The teacher asks students to tell her what they did to solve a problem but does not ask them to explain their reasoning.

● There is some questioning observed during the lesson that enables students to respond orally. However, the teacher provides most of the discussion and examples.

● The teacher asks questions and checks for understanding of the students who offer an answer but not for other students.</td><td>Examples:
●The teacher asks for thumbs up/thumbs down. This is done throughout the lesson, but this is the only means students have for communicating their understanding.

● The teacher asks students for answers, but does not ever ask, “How do you know?”

● The teacher calls on students, but feeds them the answers to a degree that it isn’t clear how much students are able to answer on their own.</td></tr>
</table>

<table>
<tr><td colspan="3">Item 14-The teacher encourages students to use mathematical vocabulary and/or symbols throughout the lesson.
This item assesses whether the teacher provides opportunities for students to be actively engaged with the terminology and symbols that are important to the subject of mathematics. Opportunities to respond should occur frequently throughout the lesson and may be aloud or in writing.</td></tr>
<tr><th>Implemented-3</th><th>Partially Implemented-2</th><th>Not Implemented-1</th></tr>
<tr><td>The teacher encourages students to use mathematical vocabulary and/or symbols throughout the lesson.</td><td>The teacher encourages students to use mathematical vocabulary and/or symbols but not consistently throughout the lesson.</td><td>The teacher does not encourage students to use mathematical vocabulary and/or symbols.</td></tr>
<tr><td>Examples:
● The teacher defines the term “area” and asks students to repeat the term. Throughout the lesson, the teacher encourages students to use the term “area” accurately and appropriately.

● The teacher consistently prompts students to answer “10 ones” instead of just saying “ten.”</td><td>Examples:
● The teacher defines a term “area” and asks students to repeat the term. Though the teacher compliments students who use the term throughout the lesson, she does not consistently ask students to use the term when appropriate.

● The teacher sometimes accepts an answer of “10” instead of “10 ones.”</td><td>Examples:
● The teacher defines a term “area” but never asks students to use it.

● The teacher accepts an answer of “10” instead of “10 ones” regularly.</td></tr>
</table>

Item 15-The teacher provides students with practice adequate to supporting the development of understanding of the procedure. This item assesses whether students have the opportunity to practice the procedure that is the focus of the lesson. To support understanding, students should practice with a range of examples that is appropriate for their skill level. A range of examples can include may include number sets with increasing difficulty, as appropriate. The examples may also include translating between representations. Also, the tasks that are involved in practice should reinforce connections and employ the visual representations that were included in the lesson.		
Implemented-3	**Partially Implemented-2**	**Not Implemented-1**
The teacher provides students with practice **adequate** to supporting the development of understanding of the procedure.	The teacher provides students with practice **somewhat adequate** to supporting the development of understanding of the procedure.	The teacher provides students with practice **inadequate** to supporting the development of understanding the procedure.
Examples: ● After a demonstration on using an area model to multiply 2 digit numbers, students practice with a worksheet with several problems that provides space and prompts for constructing an area models. ● Students begin guided practice but are struggling due to complexity of the task. The teacher adjusts and provides simpler practice that allows students to develop understanding and be successful with the procedure.	Examples: ● After a demonstration on using an area model to multiply 2 digit numbers, students practice with a worksheet that only includes two opportunities to apply the strategy. Despite time for more, the teacher directs students who finish early to alternative activities. ●Students begin guided practice but are struggling due to complexity of the task. After growing confusion, the teacher adjusts and provides scaffolds that support their ability to complete the task as given. The scaffolding only supports getting the right answer, and therefore practice that enhances understanding of the procedure is minimal.	Examples: ● Students are only given an opportunity to practice with one example during the class. ● The teacher gives a worksheet that includes number sets more complicated than those presented earlier in the lesson. ● The teacher gives a worksheet with number sets that are too simple. Students finish quickly and do not have an opportunity to transfer the skill to new situations. ● Rather than providing scaffolds when students are having difficulty, the teacher tells the students what numbers go in the blanks on their worksheets.

Item 16-The teacher uses questions and prompts related to visual representations, strategies, rules, and/or application of memory techniques to encourage students to develop independence. This item examines questions and prompts used by the teacher as guidance as students work with the procedure. Guidance is an interaction between the teacher and students. The teacher provides students with adequate support to process or reason about a given question or task. The types of questions or prompts needed may vary depending upon the complexity and nature of the question or task and the students' current level of understanding and should be adjusted accordingly. The goal is to help students use representations, strategies, rules,and memory techniques as backup strategies and to monitor and check their work.		
Implemented-3	**Partially Implemented-2**	**Not Implemented-1**
The teacher uses questions and prompts related to visual representations, strategies, rules, and/or application of memory techniques to encourage students to develop independence.	The teacher does **not effectively** use questions and prompts related to visual representations, strategies, rules, and/or application of memory techniques to encourage students to develop independence.	The teacher does **not** use questions and prompts related to visual representations, strategies, rules, and/or application of memory techniques to encourage students to develop independence.
Examples: ● The teacher reminds students they can use a number line or decomposing strategy for a multiplication fact when they encounter facts they do not have memorized. ● The teacher frequently asks students for the components of a mnemonic device when performing a multi-step procedure. ● The teacher frequently prompts students in using a visual representation to solve multi-digit multiplication. These prompts include	Examples: ● When students encounter multiplication facts they have not memorized, the teacher asks, "What's a strategy you know?" Students are unable to think of one. The teacher does not provide further more specific prompts. ● The teacher provides a mnemonic device for remembering the steps of a multi-step procedure but does not consistently prompt students to use it to monitor their own progress through a problem. Several times the teacher tells students which step they forgot.	Examples: ● When students encounter multiplication facts they have not memorized, rather than prompting students to use a strategy, the teacher gives the answer. ● The teacher does not provide any supports for remembering the steps of a multi-step procedure. ● The teacher does not question or prompt students to use a mnemonic for solving multi-digit multiplication problems.

discussion of the representation as a tool for students to approach new problems.	● The teacher sometimes prompts students to use the visual representation to solve multi-digit multiplication but does not emphasize it as a tool students can rely on to work through a problem.	

Item 17-Feedback is consistently linked to mathematical reasoning and concepts.
This item evaluates the focus of feedback. Feedback may include specific information about reasoning, processes, or calculations. Feedback can take the form of correction, suggestion, prompting, cueing or reinforcing and affirming. Both the nature and complexity of the task and the lesson goal need to be taken into consideration. Affirmative or corrective feedback that is immediate can reinforce accuracy, prevent misconceptions, or ensure accurate practice. Somewhat delayed feedback that allows the students time to think through the steps of a complex process may also be appropriate and support confidence, independence and self-regulation. Opportunities for specific feedback on reasoning or concepts may occur with practice of new tasks, after an error, or reinforcing correct thinking. When misconceptions arise, feedback should identify the incorrect reasoning, clarify the correct reasoning, and reinforce students' use of the correct reasoning.

Implemented-3	**Partially Implemented-2**	**Not Implemented-1**
Feedback is **consistently** linked to mathematical reasoning and concepts.	Feedback is **not consistently** linked to mathematical reasoning and concepts.	There is **no** feedback, OR feedback is **not** linked to mathematical reasoning and concepts.
Examples: ● When the student misses 6 x 2, the teacher gives corrective feedback with a model and describes it as groups and size of group. ● Students are successful with the tasks. The teacher takes several opportunities to give informative feedback such as, "Good. You saw that there were five parts in a whole, so you took out groups of five."	Examples: ● Feedback is specific and informative when students make a mistake, but affirmative feedback is not specific. For example, a student models 6 + 2 instead of 6 x 2. The teacher gives corrective feedback. When the student is successful with the next problem of 4 x 3, the teacher' just says, "You got it." The teacher could say, "Good. You knew that you were modeling groups and size of group."	Examples: ● The teacher consistently only tells students if they are right or wrong. If they are wrong, the teacher says, "Read it again." ● The teacher simply states that students are correct or prompts students to perform steps in the procedure. The teacher does not discuss the meaning of the steps. ● Students are successful with the tasks. The teacher only tells students, "Good job."

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